On the Zeros of a Polynomial in a Given Domain

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Abstract In this paper we obtain results concerning the bound for the number of zeros for the polynomial \( p(z) \) which generalizes well known result due to A.Ebadian, M.Bidkham and M.Eshaghi Gordji [Number of zeros of a polynomial in a given domain, Tamkang Jour. of Mathematics, Vol 42, No.4,(2011), 531-536] and also zeros of a polynomial in a given domain, Tamkang Jour, of India.

1. Introduction

Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that

\[
a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0
\]

then according to a well known result of Enstrom and Kakeya, the polynomial \( p(z) \) does not vanish in \( |z| > 1 \). Concerning the number of zeros of the polynomial in the region \( |z| \leq \frac{1}{2} \) the following result is due to Mohammad [1].

**Theorem A.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that

\[
a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0
\]

then the number of zeros of \( p(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed

\[
1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}
\]

Bidkham and Dewan [2] generalized the Theorem A for different class of polynomials and proved the following.

**Theorem B.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that

\[
a_n \leq a_{n-1} \leq \cdots \leq a_{k+1} \leq a_k \geq a_{k-1} \geq \cdots \geq a_0
\]

for some \( k \), \( 0 \leq k \leq n \), then the number of zeros of \( p(z) \) in \( |z| \leq \frac{R}{2} \), \( R > 0 \), does not exceed

\[
1 + \frac{1}{\log 2} \log \left( \log \frac{|a_n| + |a_0| + R^n (a_k - a_0)}{|a_0|} \right)
\]

for \( R \geq 1 \) (1.1)

and

\[
1 + \frac{1}{\log 2} \log \left( \log \frac{|a_n| + |a_0| + R^n (a_k - a_0)}{|a_0|} \right)
\]

for \( R \leq 1 \) (1.2)

**Theorem C.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with complex coefficients . If for some real \( \beta \), \( |\arg a_j - \beta| \leq \frac{\pi}{2} \), \( 0 \leq j \leq n \) and for some \( 0 < t \leq 1 \),

\[
|a_0| \leq t|a_1| \leq \cdots \leq t^k |a_k| \geq t^{k+1} |a_{k+1}| \geq \cdots \geq t^n |a_n|,
\]

\( o \leq k \leq n \)

then the number of zeros of \( p(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed

\[
1 + \frac{1}{\log 2} \log \left( \log \frac{R^{k+1} |a_k| + 2t^{n+1} |a_n| (|a_0|) R^n (|a_k| - |a_0|)}{|a_0|} \right)
\]

(1.3)

A.Ebadian, M.Bidkham and M.Eshaghi Gordji [3] generalizes Theorem C and proved the following results.

**Theorem D.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that

\[
a_n \leq a_{n-1} \leq \cdots \leq a_{k+1} \leq a_k \geq a_{k-1} \geq \cdots \geq a_0
\]

for some \( k \), \( 0 \leq k \leq n \), then the number of zeros of \( p(z) \) in \( |z| \leq \frac{R}{2} \), \( R > 0 \), does not exceed

\[
1 + \frac{1}{\log 2} \log \left( \log \frac{|a_n| + |a_0| + R^k (a_k - a_0) + R^n (a_k - a_0)}{|a_0|} \right)
\]

for \( R \geq 1 \) (1.4)

\[
1 + \frac{1}{\log 2} \log \left( \log \frac{|a_n| + |a_0| + R^k (a_k - a_0) + R^n (a_k - a_0)}{|a_0|} \right)
\]

for \( R \leq 1 \) (1.5)

**Theorem E.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with complex coefficients . If for some real \( \beta \), \( |\arg a_j - \beta| \leq \frac{\pi}{2} \), \( 0 \leq j \leq n \) and for some \( R > 0 \),

\[
|a_0| \leq R |a_1| \leq \cdots \leq R^k |a_k| \geq R^{k+1} |a_{k+1}| \geq \cdots \geq R^n |a_n|, \quad o \leq k \leq n
\]

then the number of zeros of \( p(z) \) in \( |z| \leq \frac{R}{2} \), \( R > 0 \) does not exceed

\[
1 + \frac{1}{\log 2} \log \left( \log \frac{2 R^{k+1} |a_k| + 2 R^n |a_n| (|a_0|) R^n (|a_k| - |a_0|)}{|a_0|} \right)
\]

(1.6)

In this paper we improve Theorem D and Theorem E for polynomials with real and complex coefficients. More
Theorem 1 Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that
\[
a_n \leq a_{n-1} \leq \cdots \leq a_{k+1} \leq a_k \geq a_{k-1} \geq \cdots \geq a_0
\]
for some \( k, 0 \leq k \leq n \), then the number of zeros of \( p(z) \) in \( |z| \leq R \delta, \delta > 0, \) and \( 0 < \delta < 1 \) does not exceed
\[
\frac{1}{\log \delta} \log \left( \frac{M}{|f(0)|} \right)
\]
and
\[
\frac{1}{\log \delta} \log \left( \frac{M}{|a_0|} \right)
\]
for \( R \geq 1 \) and \( R \leq 1 \), respectively.

Remark 1.1 If we choose \( R = 1 \), \( \delta = \frac{1}{2} \), Theorem 1 reduces to Theorem D.

Remark 2.1. If \( f(0) \neq 0 \) and \( f(z) \leq M \) in \( |z| \leq 1 \), then see (14, pp 171) the number of zeros of \( f(z) \) in \( |z| \leq \delta, 0 < \delta < 1 \) does not exceed
\[
\frac{1}{\log \delta} \log \left( \frac{M}{|f(0)|} \right)
\]
Lemma 2.2. Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with such that \( \arg a_j - \beta \leq \alpha \leq \frac{\pi}{2}, 0 \leq j \leq n \) for some real \( \beta \), then for some \( t > 0 \),
\[
|t a_j - a_{j-1}| \leq (t|a_j| - |a_{j-1}|) \cos \alpha + (t|a_j| + |a_{j-1}|) \sin \alpha
\]

3. Proof of the Theorems

Proof of Theorem 1. Consider the polynomial
\[
g(z) = (1 - z)p(z)
\]
where \( p(z) = \sum_{j=0}^{n} (a_j - a_{j-1}) z^j \).

For \( |z| \leq R \), we have
\[
|g(z)| \leq |a_n| |R^{n+1} + |a_0|
\]
and
\[
|g(z)| \leq |a_n| |R^{n+1} + |a_0| + |R| |a_n| + |a_0| + |R|^{n+1} |a_0| - |a_0|\)
\]
for \( R \geq 1 \) and \( R \leq 1 \), respectively.

Which further imply
\[
\frac{|g(z)|}{|g(0)|} \leq \frac{|a_n| |R^{n+1} + |a_0| + |R| |a_n| + |a_0| + |R|^{n+1} |a_0| - |a_0|\)}{|a_0|}
\]
for \( R \geq 1 \)

And
\[
\frac{|g(z)|}{|g(0)|} \leq \frac{|a_n| |R^{n+1} + |a_0| + |R| |a_n| + |a_0| + |R|^{n+1} |a_0| - |a_0|\)}{|a_0|}
\]
for \( R \leq 1 \)

Applying Lemma 2.1 to \((z)\), we get the number of zeros of \( g(z) \) in \( |z| \leq \delta R \), does not exceed
\[
\frac{1}{\log \delta} \log \left( \frac{M}{|f(0)|} \right)
\]
and
\[
\frac{1}{\log \delta} \log \left( \frac{M}{|a_0|} \right)
\]
for \( R \geq 1 \) and \( R \leq 1 \), respectively.

As the number of zeros of \( p(z) \) in \( |z| \leq \delta R \), does not exceed the number of zeros of \( g(z) \) in \( |z| \leq \delta R \), the theorem follows.

Proof of Theorem 2 Consider
\[
F(z) = (R - z)p(z)
\]
\[-a_n z^{n+1} + Ra_0 + \sum_{j=1}^{n} (Ra_j - a_{j-1}) z^j\]

For \(|z| \leq R\), we have

\[|F(z)| \leq |a_n| R^{n+1} + R|a_0|\]

\[+ \sum_{j=1}^{n} (R|a_j| - |a_{j-1}|) R^j \cos \alpha\]

\[+ \sum_{j=1}^{n} (R|a_j| + |a_{j-1}|) R^j \sin \alpha\]

\[= |a_n| R^{n+1} + R|a_0|\]

\[+ \sum_{j=1}^{k} (R|a_j| - |a_{j-1}|) R^j \cos \alpha\]

\[+ \sum_{j=k+1}^{n} (|a_{j-1}| + R|a_j|) R^j \cos \alpha\]

\[+ \sum_{j=1}^{n} (R|a_j| + |a_{j-1}|) R^j \sin \alpha\]

\[= 2R^{k+1}|a_k| \cos \alpha\]

\[+ 2R \sin \alpha \sum_{j=0}^{n} R^j |a_j|\]

\[\leq 2R^{k+1}|a_k| \cos \alpha\]

\[+ 2R \sin \alpha \sum_{j=0}^{n} R^j |a_j|\]

\[\leq 2R^{k+1}|a_k| \cos \alpha\]

Further proceeding on the same lines of Theorem 1, the proof of Theorem 2 can be completed.

**REFERENCES**


