Generalization of Two Types of Improper Integrals

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Abstract This article uses the mathematical software Maple for the auxiliary tool to study two types of improper integrals. We can obtain the infinite series forms of these two types of integrals by using geometric series, differentiation term by term, and differentiation with respect to a parameter. On the other hand, we provide some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

Keywords Improper Integrals, Infinite Series Forms, Geometric Series, Differentiation Term by Term, Differentiation with Respect to a Parameter, Maple

1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. As for the instructions and operations of Maple, we can refer to [1-7].

In [8], there are the following formulas of two types of improper integrals

$$\int_0^{\infty} \frac{\sin ax}{\sinh bx} dx = \frac{\pi}{2b} \tanh \frac{a\pi}{2b} \quad (1)$$

$$\int_0^{\infty} \frac{\cos ax}{\cosh bx} dx = \frac{\pi}{2b} \operatorname{sech} \frac{a\pi}{2b} \quad (2)$$

where a, b are real numbers, and $a \neq 0, b > 0$. In this study, we generalize the above improper integrals (1) and (2) to the following two types of improper integrals

$$\int_0^{\infty} \frac{x^n \sin\left(ax + \frac{n\pi}{2}\right)}{\sinh bx} dx \quad (3)$$

$$\int_0^{\infty} \frac{x^n \cos\left(ax + \frac{n\pi}{2}\right)}{\cosh bx} dx \quad (4)$$

where a, b are real numbers, $a \neq 0, b > 0$, and n is any non-negative integer. We can obtain the infinite series forms of these two types of improper integrals by using geometric series, differentiation term by term, and differentiation with respect to a parameter; these are the major results of this study (i.e., Theorems 1 and 2). The study of related integral problems can refer to [9-29]. On the other hand, we propose four examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce three important theorems used in this study.

2.1. Geometric Series

$$\frac{1}{1+u} = \sum_{k=0}^{\infty} (-1)^k u^k,$$

where u is a real number, and $|u| < 1$.

2.2. Differentiation Term by Term ([30])

For all non-negative integers k , if the functions $g_k : (a, b) \rightarrow R$ satisfy the following three conditions : (i) there exists a point $x_0 \in (a, b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a, b) , (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a, b) . Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a, b) . Moreover, its derivative $\frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

2.3. Differentiation with Respect to A Parameter ([31])

Suppose c, d, λ, β are real numbers and the function $f(a, x)$ is defined on $[c, d] \times [\lambda, \beta]$. If $f(a, x)$ and its partial derivative $\frac{\partial f}{\partial a}(a, x)$ are continuous functions on $[c, d] \times [\lambda, \beta]$. Then $F(a) = \int_{\lambda}^{\beta} f(a, x) dx$ is differentiable on the open interval (c, d) , and its derivative $\frac{d}{da} F(a) = \int_{\lambda}^{\beta} \frac{\partial f}{\partial a}(a, x) dx$ for all $a \in (c, d)$.

The following is the first result in this study, we determine the infinite series form of the improper integral (3).

2.4. Theorem 1

Suppose a, b are real numbers, $a \neq 0$, $b > 0$, and n is any non-negative integer.

Case (A): If $a > 0$. Then the improper integral

$$\begin{aligned} & \int_0^{\infty} \frac{x^n \sin\left(ax + \frac{n\pi}{2}\right)}{\sinh bx} dx \\ &= \frac{(-1)^n \pi^{n+1}}{b^{n+1}} \cdot \sum_{k=0}^{\infty} (-1)^k k^n \exp\left(-\frac{ak\pi}{b}\right) \end{aligned} \quad (5)$$

Case (B): If $a < 0$. Then

$$\begin{aligned} & \int_0^{\infty} \frac{x^n \sin\left(ax + \frac{n\pi}{2}\right)}{\sinh bx} dx \\ &= -\frac{\pi^{n+1}}{b^{n+1}} \cdot \sum_{k=0}^{\infty} (-1)^k k^n \exp\left(\frac{ak\pi}{b}\right) \end{aligned} \quad (6)$$

2.4.1. Proof

Case (A): If $a > 0$. Because

$$\begin{aligned} & \int_0^{\infty} \frac{\sin ax}{\sinh bx} dx \\ &= \frac{\pi}{2b} \tanh \frac{a\pi}{2b} \quad (\text{By (1)}) \\ &= \frac{\pi}{2b} \cdot \frac{\exp\left(\frac{a\pi}{2b}\right) - \exp\left(-\frac{a\pi}{2b}\right)}{\exp\left(\frac{a\pi}{2b}\right) + \exp\left(-\frac{a\pi}{2b}\right)} \\ &= \frac{\pi}{2b} \cdot \frac{1 - \exp\left(-\frac{a\pi}{b}\right)}{1 + \exp\left(-\frac{a\pi}{b}\right)} \\ &= \frac{\pi}{2b} \cdot \left[-1 + \frac{2}{1 + \exp\left(-\frac{a\pi}{b}\right)} \right] \\ &= \frac{\pi}{2b} \cdot \left[-1 + 2 \cdot \sum_{k=0}^{\infty} (-1)^k \exp\left(-\frac{ak\pi}{b}\right) \right] \\ & \quad (\text{Using geometric series}) \quad (7) \\ &= -\frac{\pi}{2b} + \frac{\pi}{b} \cdot \sum_{k=0}^{\infty} (-1)^k \exp\left(-\frac{ak\pi}{b}\right) \end{aligned} \quad (8)$$

By differentiation term by term and differentiation with respect to a parameter, differentiating n -times with respect to a on both sides of (8), we obtain

$$\begin{aligned} & \int_0^{\infty} \frac{x^n \sin\left(ax + \frac{n\pi}{2}\right)}{\sinh bx} dx \\ &= \frac{(-1)^n \pi^{n+1}}{b^{n+1}} \cdot \sum_{k=0}^{\infty} (-1)^k k^n \exp\left(-\frac{ak\pi}{b}\right) \end{aligned}$$

Case (B): If $a < 0$. Because

$$\begin{aligned} & \int_0^{\infty} \frac{\sin ax}{\sinh bx} dx \\ &= \frac{\pi}{2b} \tanh \frac{a\pi}{2b} \\ &= -\frac{\pi}{2b} \tanh \left(\frac{-a\pi}{2b} \right) \end{aligned}$$

$$\begin{aligned}
&= -\frac{\pi}{2b} \cdot \left[-1 + 2 \cdot \sum_{k=0}^{\infty} (-1)^k \exp\left(\frac{ak\pi}{b}\right) \right] \quad (\text{By (7)}) \\
&= \frac{\pi}{2b} - \frac{\pi}{b} \cdot \sum_{k=0}^{\infty} (-1)^k \exp\left(\frac{ak\pi}{b}\right) \quad (9)
\end{aligned}$$

Also, by differentiation term by term and differentiation with respect to a parameter, differentiating n -times with respect to a on both sides of (9), we obtain

$$\begin{aligned}
&\int_0^{\infty} \frac{x^n \sin\left(ax + \frac{n\pi}{2}\right)}{\sinh bx} dx \\
&= -\frac{\pi^{n+1}}{b^{n+1}} \cdot \sum_{k=0}^{\infty} (-1)^k k^n \exp\left(\frac{ak\pi}{b}\right) \quad \text{q.e.d.}
\end{aligned}$$

The following is the second major result in this paper, we find the infinite series form of the improper integral (4).

2.5. Theorem 2

Let the assumptions be the same as Theorem 1.

Case (A): If $a > 0$. Then the improper integral

$$\begin{aligned}
&\int_0^{\infty} \frac{x^n \cos\left(ax + \frac{n\pi}{2}\right)}{\cosh bx} dx \\
&= \frac{(-1)^n \pi^{n+1}}{2^n b^{n+1}} \cdot \sum_{k=0}^{\infty} (-1)^k (2k+1)^n \exp\left(-\frac{a(2k+1)\pi}{2b}\right) \quad (10)
\end{aligned}$$

Case (B): If $a < 0$. Then

$$\begin{aligned}
&\int_0^{\infty} \frac{x^n \cos\left(ax + \frac{n\pi}{2}\right)}{\cosh bx} dx \\
&= \frac{\pi^{n+1}}{2^n b^{n+1}} \cdot \sum_{k=0}^{\infty} (-1)^k (2k+1)^n \exp\left(\frac{a(2k+1)\pi}{2b}\right) \quad (11)
\end{aligned}$$

2.5.1. Proof

Case (A): If $a > 0$. Because

$$\begin{aligned}
&\int_0^{\infty} \frac{\cos ax}{\cosh bx} dx \\
&= \frac{\pi}{2b} \operatorname{sech} \frac{a\pi}{2b} \quad (\text{By (2)}) \\
&= \frac{\pi}{b} \cdot \frac{1}{\exp\left(\frac{a\pi}{2b}\right) + \exp\left(-\frac{a\pi}{2b}\right)} \\
&= \frac{\pi}{b} \cdot \exp\left(-\frac{a\pi}{2b}\right) \cdot \frac{1}{1 + \exp\left(-\frac{a\pi}{b}\right)} \\
&= \frac{\pi}{b} \cdot \exp\left(-\frac{a\pi}{2b}\right) \cdot \sum_{k=0}^{\infty} (-1)^k \exp\left(-\frac{ak\pi}{b}\right) \quad (\text{By geometric series})
\end{aligned}$$

$$= \frac{\pi}{b} \cdot \sum_{k=0}^{\infty} (-1)^k \exp\left(-\frac{a(2k+1)\pi}{2b}\right) \quad (12)$$

Using differentiation term by term and differentiation with respect to a parameter, differentiating n -times with respect to a on both sides of (12), we have

$$\begin{aligned}
&\int_0^{\infty} \frac{x^n \cos\left(ax + \frac{n\pi}{2}\right)}{\cosh bx} dx \\
&= \frac{(-1)^n \pi^{n+1}}{2^n b^{n+1}} \cdot \sum_{k=0}^{\infty} (-1)^k (2k+1)^n \exp\left(-\frac{a(2k+1)\pi}{2b}\right)
\end{aligned}$$

Case (B): If $a < 0$. Because

$$\begin{aligned}
&\int_0^{\infty} \frac{\cos ax}{\cosh bx} dx \\
&= \frac{\pi}{2b} \operatorname{sech} \frac{a\pi}{2b} \\
&= \frac{\pi}{2b} \left(\operatorname{sech} \frac{-a\pi}{2b} \right) \\
&= \frac{\pi}{b} \cdot \sum_{k=0}^{\infty} (-1)^k \exp\left(\frac{a(2k+1)\pi}{2b}\right) \quad (\text{Using (12)}) \quad (13)
\end{aligned}$$

Also, by differentiation term by term and differentiation with respect to a parameter, differentiating n -times with respect to a on both sides of (13), we obtain

$$\begin{aligned}
&\int_0^{\infty} \frac{x^n \cos\left(ax + \frac{n\pi}{2}\right)}{\cosh bx} dx \\
&= \frac{\pi^{n+1}}{2^n b^{n+1}} \cdot \sum_{k=0}^{\infty} (-1)^k (2k+1)^n \exp\left(\frac{a(2k+1)\pi}{2b}\right) \quad \text{q.e.d.}
\end{aligned}$$

3. Examples

In the following, for the two types of improper integrals in this study, we provide four examples and use Theorems 1, 2 to determine their infinite series forms. On the other hand, we employ Maple to calculate the approximations of these improper integrals and their solutions for verifying our answers.

3.1. Example 1

In Theorem 1, let $a = 6, b = 2, n = 8$. By Case (A) of Theorem 1, we obtain the improper integral

$$\int_0^{\infty} \frac{x^8 \sin 6x}{\sinh 2x} dx = \frac{\pi^9}{2^9} \cdot \sum_{k=0}^{\infty} (-1)^k k^8 \exp(-3k\pi) \quad (14)$$

Next, we use Maple to verify the correctness of (14).

```
>evalf(int(x^8*sin(6*x)/sinh(2*x),x=0..infinity),14);
```

```
-0.0046015343323402
```

```
>evalf(Pi^9/2^9*sum((-1)^k*k^8*exp(-3*k*Pi),k=0..infinity),14);
```

infinity),14);

-0.0046015343323412

6*k+3)*Pi/8),k=0..infinity),18);

-0.000200107507162211670

3.2. Example 2

In Theorem 1, taking $a = -4, b = 5, n = 9$. Using Case (B) of Theorem 1, we can determine the improper integral

$$\int_0^{\infty} \frac{x^9 \cos 4x}{\sinh 5x} dx = -\frac{\pi^{10}}{5^{10}} \cdot \sum_{k=0}^{\infty} (-1)^k k^9 \exp\left(-\frac{4k\pi}{5}\right) \quad (15)$$

We also use Maple to verify the correctness of (15).

>evalf(int(x^9*cos(4*x)/sinh(5*x),x=0..infinity),14);

0.0056005919942804

>evalf(-Pi^10/5^10*sum((-1)^k*k^9*exp(-4*k*Pi/5),k=0..infinity),14);

0.0056005919942808

3.3. Example 3

In Theorem 2, let $a = 7, b = 4, n = 12$. By Case (A) of Theorem 2, we can evaluate the improper integral

$$\begin{aligned} & \int_0^{\infty} \frac{x^{12} \cos 7x}{\cosh 4x} dx \\ &= \frac{\pi^{13}}{2^{12} \cdot 4^{13}} \cdot \sum_{k=0}^{\infty} (-1)^k (2k+1)^{12} \exp\left(-\frac{(14k+7)\pi}{8}\right) \quad (16) \end{aligned}$$

Using Maple to verify the correctness of (16) as follows:

>evalf(int(x^12*cos(7*x)/cosh(4*x),x=0..infinity),14);

0.00070639829097714

>evalf(Pi^13/(2^12*4^13)*sum((-1)^k*(2*k+1)^12*exp(-(14*k+7)*Pi/8),k=0..infinity),14);

0.00070639829097759

3.4. Example 4

In Theorem 2, let $a = -6, b = 8, n = 15$. Using Case (B) of Theorem 2, we obtain the improper integral

$$\begin{aligned} & \int_0^{\infty} \frac{x^{15} \sin 6x}{\cosh 8x} dx \\ &= -\frac{\pi^{16}}{2^{15} \cdot 8^{16}} \cdot \sum_{k=0}^{\infty} (-1)^k (2k+1)^{15} \exp\left(-\frac{(6k+3)\pi}{8}\right) \quad (17) \end{aligned}$$

We also use Maple to verify the correctness of (17).

>evalf(int(x^15*sin(6*x)/cosh(8*x),x=0..infinity),18);

-0.000200107507162211911

>evalf(-Pi^16/(2^15*8^16)*sum((-1)^k*(2*k+1)^15*exp(-

4. Conclusions

As mentioned, the geometric series, the differentiation term by term, and the differentiation with respect to a parameter play significant roles in the theoretical inferences of this study. In fact, the applications of these theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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