Stone Duality on P-Rings

V. Amarendra Babu1*, P.Koteswara Rao2

1Department of Mathematics, Acharya Nagarjuna University Nagarjuna Nagar – 522 510
2Department of Commerce & Business Admn, Acharya Nagarjuna University, Nagarjuna Nagar, 522510, A.P, India
*Corresponding Author: amarendravelisela@ymail.com

Abstract For given p (= prime), a p-ring as first introduced by Mc Coy and Montgomery [2]. The concept of p-ring is an evident generalization of that of Boolean ring (p = 2). The well known result of Stone [7], each Boolean ring is isomorphically representable as a ring of classes or what is equivalent, is isomorphic with a sub ring of some direct power of Zp (field of residues mod p) and they showed that each polynomial power of Z p. The present communication concerned with a further study of p-rings. In particular we study the topological properties of p-rings and proved a Stone duality theorem.

Keywords P-Ring, Boolean Ring, Prime Ideals

1. Preliminaries

1.1. Definition [2] Suppose p is a prime number. A commutative ring R with unity in which aP = a, pa = 0 for every a R is called a p-ring.

1.2. Definition [1] Let J be a Boolean ring. Let n be an integer, n ≥ 2. By a vector partition of J, of degree n, also called a J-vector, or Boolean vector, we understand an ordered n-tuple of pair wise disjoint elements of J,

\[ b = (b_1, b_2, \ldots, b_n) \]

\[ b_i \times b_j = 0 \quad (i \neq j) \]

The components of b. Two vectors are equal if their corresponding vectors are equal i.e., \( b = c \Leftrightarrow b_i = c_i \quad i=1,2,\ldots,n \).

1.3. Definition [1] If the vector partition \( b = (b_1, b_2, \ldots, b_n) \) satisfies \( \sum b_i = b' + b'' + \cdots + b^{(n)} = 1 \) then \( b \) is called a complete vector. If \( b \) is complete vector, then it is denoted by \( b = (b_1, b'_2, \ldots, b^{(n)}) \).

1.4. Note [1] Each component of a complete J-vector is determined from the remaining components i.e., if \( b = [b_1, b_2, \ldots, b^{(n-1)}] \) then \( b_i = 1 - (b_1 + b_2 + \cdots + b^{(n-1)}) \).

There is a one to one correspondence \( [b_0, b_1, b'_2, \ldots, b^{(n-1)}] \leftrightarrow (b_1, b'_2, \ldots, b^{(n-1)}) \) between \( J[n] \) the set of all complete J-vectors of degree n and \( J^{(n-1)} \), the set of all J-vectors of degree n-1 i.e., \( J^n \cong J^{(n-1)} \).

1.5. Theorem [1]. If \( J = J_2^k \) is the finite Boolean ring possessing exactly k atoms (and hence \( 2^k \) elements) then \( J^{(0)} \), (and therefore also \( J^{(n-1)} \)) consists of \( n \) elements (= vectors).

1.6. Theorem [1] For a given Boolean ring J and a given integer n (n≥2),

1.6.1 The system \( (J[n], +, \times) \), with +,× defined by for

\[ a = [a_0, a_1, a_2, \ldots, a_n] \]

\[ b = [b_0, b_1, b_2, \ldots, b_n] \]

\[ a + b = [c_0, c_1, c_2, \ldots, c_n] \]

\[ c_i = \sum_{r+s=i (mod n)} a_r b_s \]

\[ ab = [d_0, d_1, d_2, \ldots, d_n] \] where

\[ d_i = \sum_{r+s=i (mod n)} a_r b_s \]

1.6.2 of characteristic n,

\[ na = a + a + \ldots + a = 0 \quad (a \in J^{(0)}) \]

1.6.3 For n = p-prime,

\[ a^p = a \times a \times \ldots \times a \]

\[ a = (a \in J^{(p)}) \] and hence \( J^{(p)} \) is a p-ring

1.7. Note [1] In the p-ring \( J^{(p)} \), +,×, 0,1),

\[ 2 = 1 + 1, \quad 3 = 1 + 1 + 1, \ldots, \quad n = 1 + 1 + \ldots + 1, \quad n \text{ terms} = 0 \]

1.8. Note [1] 1. In \((J^{(p)}, +,\times)\), if

\[ a = [a_0, a_1, a_2, \ldots, a_n] \]

\[ -a = [a_0, a_1, a_2, \ldots, a_n] \]

\[ a^* = 1 - a = [a_1, a_2, a_3, \ldots, a_n] \]

2. Via the 1-1 correspondence

\[ a \leftrightarrow [a^*, a] \leftrightarrow (a), \]

\[ J \cong (J[2], +,\times) \cong J^{(0)}, +,\times) \]

1.9. Definition [1] A ring is complete if the sum, and also the product of an arbitrary (not necessarily denumerable)
subset of elements of the ring is defined and is an element of the ring and both associativity and distributivity holds for these general sums and products.

1.10. Theorem [1] If $J$ is a complete Boolean ring, then the vector ring $\langle J^{[p]} \rangle$ over $J$ is a complete ring.

1.11. Theorem [1] (Normal Representation Theorem) In a p-ring $R = (R, +, \times)$, each $a \in R$, may be decomposed in one and only one way in the “normal idempotent form”

$$a = a_1^2 + 2a_2^2 + 3a_3^2 + \cdots + (p-1)^2 a_{p-1}$$

in which the “normal components”

$$a_k = (a_i, a_2, a_3, \ldots, a_{p-1})$$

of the element $a$ are idempotent elements of $R$ and pair wise disjoint.

Here the coefficients on the right side of $a_k$ are to be taken mod $p$.

1.12. Note [1] Suppose $R = (R, +, \times)$ is a p-ring. The set $I$, consisting of $p$ “integers” of $R$,

\[ I = \{0, 1, 2, \ldots, p-1\} \]

forms a sub ring of $R$ and $I \cong Z_p$.

1.13. Theorem [1] If $R = (R, +, \times)$ is a p-ring and let $J = (J^{[p]}, +, \times)$ be idempotent Boolean-sub ring of $R$. If $a = a_1 + 2a_2 + 3a_3 + \cdots + (p-1)^2 a_{p-1}$, then $(J, +, \times)$ is a complete Boolean algebra.

2. Prime Ideal Spaces

2.1 Definition: A non-empty subset $I$ of a p-ring $R$ is called an ideal if

- $(a, b, c \in I) \Rightarrow a-b, c-d \in I$
- $(a \in I, r \in R) \Rightarrow ra \in I$
- $(a \in I) \Rightarrow a \neq 0$

2.2 Theorem [3] Suppose $I$ is an ideal in a p-ring $R$. Define $a \mapsto b$ iff $a-b \in I$. Then $\theta$ is a congruence relation on $R$ and $I = \emptyset$.

Proof: Suppose $I$ is an ideal in a p-ring $R$. Define $0 \in R$ by $a \mapsto b$ iff $a-b \in I$. $0 \in I \Rightarrow a \in R \forall a \in R$

• $\theta$ is reflexive
• $\theta$ is transitive
• $\theta$ is an equivalence relation

2.3. Theorem: $\theta$ is a congruence relation on $R$. Claim: $I = \emptyset(0)$

- $a \in I \Rightarrow a \in I$
- $a \in I \Rightarrow a \theta 0$
- $a \in I \Rightarrow a \in I(0)$
- $I = \emptyset(0)$

2.4 Definition. An ideal $P$ of a p-ring $R$ is a prime ideal if $ab \in P \Rightarrow a \in P$ or $b \in P$.

2.5. Theorem. Let $\theta$ be a congruence relation on a p-ring $R$. Then

\[ \sum_{r+s = k \pmod {p}} a b = \prod_{k} (k^2 a+b)^{p-1} \]
R/0 = {θ(a)/a ∈ R} is a p-ring.
Proof: Suppose R is p-ring and 0 is a congruence relation on R.

Claim: R/0 = {θ(a)/a ∈ R} is a p-ring.

*: R is a Commutative ring, so R/0 is a commutative ring.

Suppose 0(a) ∈ R/0.
0(a)² = 0(a²) = 0(a).
p0(a) = 0(pa) = 0(0).
∴ R/0 is a p-ring.

2.6 Theorem [5]: f : R → R is a homomorphism of p-rings then Ker f is an ideal of R.

Proof: Ker f = {a ∈ R/f(a) = 0}.
Let a, b ∈ Ker f ⇒ f(a) = 0, f(b) = 0.
f(a + b) = f(a) + f(b) = 0 + 0 = 0.
∴ a + b ∈ Ker f.
Let a ∈ Ker f, r ∈ R.
⇒ f(a) = 0 and r ∈ R.
Consider f(ra) = rf(a)
= r.0
= 0.
∴ ra ∈ Ker f.
∴ Ker f is an ideal of R.

2.7 Theorem [5]: Every ideal of a p-ring R is the kernel of a p-ring homomorphism.

Proof: Suppose I is an ideal of R.
Suppose 0 is a congruence relation on R by a0b if a-b ∈ I.
Clearly I = {0}.
Define f : R → R/0 by
f(a) = (θ(a) ∀ a ∈ R).
Clearly f is a ring homomorphism.
Ker f = {a ∈ R/f(a) = 0}
{a ∈ R/θ(a) = 0 = 0(0)}
{a ∈ R/θ(a) = 0(0)}
= 0(0)
= 0
= I.

2.8 Theorem: Let R be a p-ring and J = J(R) be its Boolean ring of idempotents of R, J be a congruence relation on J. The relation φ on R defined by φ(a) b ⇔ a, b are idempotent components of a, b respectively, is a Congruence on R.

Proof: Suppose φ is a congruence relation on J = J(R).
Define φ on R by φ(a) b ⇔ a, b where a, b are idempotent components of a, b respectively (1.11 Theorem)

Claim: φ is a congruence relation on R.
Suppose a ∈ R.
*: aφ b ⇒ a, b, aφ b, aφ b, aφ b, aφ b, aφ b, aφ b
∴ aφ b ⇔ a, b.
Suppose aφ b, bφ c
∴ aφ b, bφ c, aφ b, aφ b, aφ b, aφ b, aφ b, aφ b
∴ aφ c, aφ c, aφ c, aφ c
∴ aφ c, aφ c, aφ c, aφ c
∴ aφ c

:. φ is an equivalence relation on R.
Suppose aφ b, cφ d.
⇒ a, b, c, d.
⇒ (a + c) φ (b + d), a, c, b, d
⇒ (a + c) φ (b + d), acφ bd
∴ φ is a congruence relation on R.

2.9 Theorem: If φ is a Congruence on a p-ring R, then

φ = φ (Xφ J) is a Congruence on J and φ = φ.

Proof: Claim: φ = φ (Xφ J) is Congruence on J and φ = φ.

Let a ∈ J, a ∈ R
⇒ (aφ a) ∩ (Xφ J)
⇒ a φ a ∀ a ∈ J
∴ φ is reflexive.

aφ b ⇒ bφ a clearly.
aφ b, bφ c ⇒ aφ c.

Suppose aφ b, cφ d
⇒ aφ b, bφ d & a, b, c, d ∈ J
⇒ (a+b) φ (c+d), ab φ cd and a+b, c+d, ab, cd ∈ J
∴ (a+b) φ (c+d), ab φ cd.
∴ φ is a congruence relation on J.

Claim: φ = φ.
(a,b) ∈ φ ⇔ aφ b, bφ i=1,2,......p-1
⇔ aφ b, a, b ∈ J ∀ i=1,2,......p-1
⇔ aφ b
∴ φ = φ.

2.10 Note: If φ an equivalence on a p-ring R and φ is a Congruence relation on J, then φ ≠ φ.

Eg: 3 = {0, 1, 2} is a 3-ring.
φ = {(0,0), (1,1), (2,2), (2,1), (1,2)} is an equivalence relation on 3 = {0,1,2}.

Consider φ2 = {(0, 0), (1, 1), (2, 2)}.
∴ φ2 ≠ φ.

2.11 Theorem: Let R be a p-ring, φ be a congruence relation on R iff
φ is a Congruence on J and φ = φ.

Proof: Suppose φ is a Congruence relation on R.
⇒ φ = φ.

Clearly φ = φ = φ.

2.12 Theorem: Every prime ideal in a p-ring R is a maximal ideal Proof: Suppose P is a prime ideal in R. Claim: P is maximal.

Let M be an ideal such that P ⊆ M ⊆ R.

Let x ∈ M − P ⇒ x ∉ P

⇒ x₁ ∈ P for at least i = 1, 2, ..., p - 1 say i = 1 i.e. x₁ ∉ P.

∴ x₁x₁ = 0 ∈ P ⇒ x₁x₁ ∈ P

⇒ x₁ ∈ P or x₁ ∈ P :

∴ x₁ ∈ P, so x₁ ∈ P

⇒ x₁ ∈ M.

∴ M = R.

Claim: I is a maximal ideal.

2.13 Theorem: An ideal I of a p-ring is maximal if and only if J is a Boolean maximal ideal

Proof: Suppose a₁ ∈ I is an ideal of J such that I ∩ J is a Boolean maximal ideal.

Let M = I ∩ J.

Then M is clearly Pk.

Claim: M = R.

Suppose M is ideal in R such that M is maximal.

Then K = M ∩ R.

Since I is maximal, K = R

∴ K = R for at least i = 1, 2, ..., p - 1

Suppose a₁ ≠ 0 for at least one k = 1, 2, ..., p - 1

Let P = (p₁) R = R.

∴ a₁ ∈ P.

Similarly, we get a₁ ∈ P.

2.14 Definition: Suppose I is an ideal of a p-ring R. For any a ∈ R, we define

Iₐ = (ba₁b₁,a₁b₁) ∩ I.

Iₐ is called a coset of R.

2.15 Theorem: If P is a prime ideal of R then

Pₐ = (b ∈ P/a₁b₁ ∈ I₁).

Proof: P is prime ideal.

For k ∈ Zₚ = {0, 1, 2, ..., p - 1},

Clearly, Pₐ = (b ∈ R/k b₁b₁ ∈ P₁, b₁ ∈ P₁, i = 1, 2, ..., p - 1).

Let a ∈ R.

Suppose a₁ P₀,a₁ P₁,a₁ P₂, ..., a₁ Pₚ₋₁.

⇒ For at least one i = 1, 2, ..., p - 1, a₁ Pᵢ and 1 a₁ P₁ and ..., (p - 2) a₁ Pₚ₋₁.

∴ 0 a₁, a₁,w ..., (p - 2) a₁, (p - 1) a₁ Pₚ₋₁.

Similarly, (p - 1) a₁ Pₚ₋₁.

Define f: R P → Zₚ = {0, 1, 2, ..., p - 1} by f(Pₖ) = k where k = 0, 1, 2, ..., p - 1.

Then f is clearly a Ring isomorphism.

∴ R P ≅ Zₚ.

2.16 Theorem: For every a ≠ 0, there is a prime ideal P such that a P.

Proof: Suppose a₁ ≠ 0 for at least one k = 1, 2, ..., p - 1.

Let P = (p₁) R = R.

Suppose a₁ ≠ 0 and P = (p₁) R is a prime ideal of R.

Then f is an epimorphism where P denotes by Pₐ.

2.17 Note: Consider the family {Rᵢ/i ∈ I} of p-rings iff for every a ∈ I, a₁ ∈ Rᵢ.

2.18 Define: Let {Rᵢ/i ∈ I} be a family of p-rings.

Define

Rᵢ = ∏ Rᵢ = {f ∈ f : I → ∪ Rᵢ, f(i) ∈ Rᵢ}.

Then Rᵢ is a p-ring with +, · by

(f + g)(a₁) = f(a₁) + g(a₁)

(fg)(a₁) = f(a₁)g(a₁), a₁ ∈ Rᵢ.

2.19 Definition: Let {Rᵢ/i ∈ I} be a family of p-rings. For every j ∈ I, define

πᵢ: ∏ Rᵢ → Rᵢ by πᵢ(a₁) = aᵢ, πᵢ is called projection map

and it is a surjective homomorphism. πᵢ is also called canonical epimorphism.

2.20 Definition: A p-ring R is called a sub direct product of family {Rᵢ/i ∈ I} of p-rings if there is a monomorphism k: R → ∏ Rᵢ such that πᵢk is an epimorphism, where πᵢ: ∏ Aᵢ → Aᵢ is the canonical epimorphism.

2.21 Theorem: A p-ring R is a sub direct product of family {Rᵢ/i ∈ I} of p-rings if

Rᵢ = R/Kᵢ, Kᵢ is an ideal and ∏ Kᵢ = {0}.

Proof: Suppose R is a sub direct product of family
\{R_i/i \in I\} of p-rings.
\[ \Rightarrow \exists \text{ a monomorphism } k: R \to \prod_{i \in I} R_i \text{ such that } \pi_i k: \]
\[ R \to R_i \text{ is an epimorphism.} \]
\[ \Rightarrow R_i \cong R/\ker \pi_i k. \]
Let \( K_i = \ker \pi_i k. \)
\( \Rightarrow K_i \) is an ideal and \( R_i \cong R/K_i. \)
To show \( \text{ \( \bigcap_{i \in I} K_i = \{0\} \) } \)
If possible \( 0 \neq a \in \bigcap_{i \in I} K_i \iff a \in K_i \forall i \in I \)
\( \iff (\pi_i k)(a) = 0 \forall i \in I \)
\( \iff \pi_i(k(a)) = 0 \forall i \in I \)
\( \iff \text{\( i \) th component of } k(a) = 0 \)
\( \iff k(a) = 0 \)
\( \iff a = 0 \)
\( \therefore \bigcap_{i \in I} K_i = \{0\}. \)
Conversely suppose that there exists a family \( \{K_i/i \in I\} \) of ideals of \( R \) such that
\( \bigcap_{i \in I} K_i = \{0\}. \)
Let \( R_i = R/K_i. \)
Define \( k: R \to \prod_{i \in I} R_i \) by
\[ k(r) = \{K_i r + i/I\}. \]
Claim: \( k \) is a monomorphism.
\[ r \in \ker k \iff k(r) = 0 \text{ (LINE 330)} \]
\( \iff K_i r = 0 \forall i \in I \)
\( \iff r \in K_i \forall i \in I \)
\( \iff r \in \bigcap_{i \in I} K_i = \{0\} \)
\( \therefore r = 0. \)
\( \therefore \ker k = \{0\} \Rightarrow k \text{ is monomorphism.} \)
Hence \( R \) is a sub direct product of family \( \{R_i/i \in I\} \) of p-rings.
2.22 Theorem: Every p-ring \( R \) is a sub direct Product of Copies of \( \mathbb{Z}_p = \{0,1,2,\ldots,(p-1)\} \)
Proof: Suppose \( R \) is a p-ring.
Suppose \( \{P_i/i \in I\} \) is the family of all prime ideals of \( R. \)
\[ \Rightarrow \bigcap_{i \in I} P_i = \{0\} \text{ (by 1.33 notes)} \]
This \( P_i \) is a sub direct product of \( \{R/P_i/i \in I\} \) of p-rings.
\( \therefore \) there is a monomorphism \( k: R \to \prod_{i \in I} P_i. \)
Since every \( P_i \) is a prime ideal, \( R/P_i \cong \mathbb{Z}_p. \)(by Theorem 2.15)
Hence \( R \) is a sub direct Product of Copies of \( \mathbb{Z}_p. \)
2.23 Notations: Suppose \( R \) is a p-ring. Let \( X \) be the set of all prime ideals of \( R. \)
Let \( a \in R, C \subseteq R. \)
\[ X_a = \{P \in X/a \notin P\}. \]
\[ X_a = \{P \in X/C \notin P\}. \]
2.24 Notes: (i) \( X_1 = \{P \in X/1 \notin P\} = X \)
(ii) \( X_0 = \{P \in X/0 \notin P\} = \phi \)
(iii) For \( a,b \in R, \)
\[ ab = \{a,by/i=1,2,\ldots,p-1, j=1,2,\ldots,p-1\} \text{ (by Theorem 1.11)} \]
2.25 Lemma: Let \( a,b \in R, C \subseteq R \)
\( i)X_{ab} = X_a \cap X_b \)
\( ii)X_C = \bigcup X \cdot \quad a \in C \quad a \in C \)
\( \therefore X_C = \bigcup X \cdot \quad a \in C \quad a \in C \)
2.26 Definition: \( R, X \) as in 2.23 Notation. From 2.24 Note (i) and (ii), \( \{X_a/a \in R\} \) forms an open base for which \( X \) is a topological space. This topological space \( X \) is called prime ideal space of \( R \) and it is denoted by Spec \( R. \)
2.27 Note: \( C \subseteq R, D \subseteq R \)
\[ \text{CAD} = \bigcup a \Delta b \text{ (see 2.24 note (iii))} \]
\[ a \in C \quad b \in D \]
2.28 Lemma: \( X_C \cap X_D = X_{CAD} \)
Proof: \( P \in X_C \cap X_D \iff P \in X_C \text{ and } P \in X_D \)
\( \iff C \subseteq P \text{ and } D \subseteq P \)
\( \iff \exists a \in C, b \in D \ni a \notin P \text{ and } b \notin P \)
\( \iff a \Delta b \subseteq P \)
\( \iff CAD \subseteq P \)
\( \therefore P \in CAD \)
\[ \therefore X_C \cap X_D = X_{CAD}. \]
2.29 Note: \( X_{a,b} = X_a \cup X_b. \)
2.30 Lemma: For every \( e \in J, X_e \) is a clopen set.
Proof: \( X_e = \{P \in X/e \notin P\} \)
\( \therefore X_e \text{ is a clopen set} \)
\[ \therefore X_e \text{ is closed set.} \]
2.31 Theorem: Suppose \( R \) is a p-ring. Then \( X, \) the Prime ideal space of \( R, \) is totally disconnected.
Proof: Suppose \( P_1, P_2 \in X \& P_1 \neq P_2 \)
\[ \Rightarrow P_1, P_2 \text{ are two distinct prime ideals of } R. \]
Let \( a \in P_1, P_2. \)
\[ \Rightarrow a \in P_1 \text{ and } a \notin P_2 \]
\[ \therefore a \in P_1 \Rightarrow a \in P_1 \forall i=1,2,\ldots,p-1 \]
\[ \therefore a \in R \text{ and } P_2 \text{ is prime ideal so } a \in P_2 \text{ for at least one } i=1,2,\ldots,p-1. \]
Suppose \( a \notin P_2. \)
\[ \therefore a \notin P_1, a \notin P_2. \]
\[ \Rightarrow a \notin P_1, a \notin P_2. \]
\[ \Rightarrow P_1 \in X_{a_1} \land P_2 \in X_{a_1} \]

Now \( X_{a_1} \land X_{a_1} = X_{a_1} \land a_1 = X_0 = \emptyset \)

And \( X_{a_1} \cup X_{a_1} = X_i = X \)

For \( P_1, P_2 \in X \) with \( P_1 \neq P_2 \), \( \exists \) two disjoint open sets \( X_{a_1} \land \) \( X_{a_2} \).

Thus \( X \) is totally disconnected and hence \( X \) is Hausdorff space.

2.32 Theorem: \( X \) is Compact.

Proof: Suppose \( \{ X_a / a \in C \} \) is a basic open cover for \( X \) where \( C \subseteq R \).

\[ \Rightarrow X = \bigcup_{a \in C} X_a \]

Suppose there is no finite sequence \( a_1, a_2, \ldots, a_n \in C \) \( \ni \) \( a_1 \lor a_2 \lor \ldots \lor a_n \neq 1 \).

Then the ideal generated by \( C \) i.e., \( \langle C \rangle \) is a proper ideal of \( R \)

\[ \Rightarrow \exists M \] a maximal ideal of \( R \) such that \( \langle C \rangle \subseteq M \)

\[ \therefore M \text{ is maximal ideal of } R \text{ and } M \text{ is a prime ideal of } R \]

\[ \therefore C \subseteq M \Rightarrow a \in M \forall a \in C \]

\[ \Rightarrow M \notin \bigcup_{a \in C} X_a \]

\[ \Rightarrow M \notin X \]

It is a contradiction.

\[ \therefore \exists a_1, a_2, \ldots, a_n \in C \ni a_1 \lor a_2 \lor \ldots \lor a_n = 1 \]

\[ \Rightarrow X = \bigcup_{a \in C} X_a = X_{a_1} \lor X_{a_2} \lor \ldots \lor X_{a_n} \]

\[ = X_{a_1} \lor X_{a_2} \lor \ldots \lor X_{a_n} \]

\[ \therefore \{ X_a / a \in C \} \text{ has a finite sub cover for } X \]

Hence \( X \) is compact.

2.33 Theorem: Suppose \( R \) is a p-ring and \( a \in R \), then

\[ X_a = \bigcup_{a \in C} X_{a_1} \lor X_{a_2} \lor \ldots \lor X_{a_n} \]

\[ = X_{a_1} \lor X_{a_2} \lor \ldots \lor X_{a_n} \]

\[ = X_{a'}, \quad e = a_1 \lor a_2 \lor \ldots \lor a_n \]

Proof: \( X_{a'} = \{ P \in X / a \notin P \} \)

\[ = \{ P \in X / a_1 \notin P \lor a_2 \notin P \lor \ldots \lor a_{n-1} \notin P \} \]

\[ = \{ P \in X / a_1 \notin P \} \cup \{ P \in X / a_2 \notin P \} \cup \ldots \cup \{ P \in X / a_{n-1} \notin P \} \]

\[ = X_{a_1} \lor X_{a_2} \lor \ldots \lor X_{a_{n-1}} \]

\[ = X_{a'} \lor a_{n-1} \]

\[ = X_{a'} \lor a_{n-1} \]

\[ \Rightarrow X_{a'} \]

REFERENCES


