Pulse Vaccination Strategy in a SVEIRS Epidemic Model with Two-Time Delay and Saturated Incidence

Gbolahan Bolarin¹*, O.M Bamigbola²

¹Mathematics and Statistics Department, School of Natural and Applied Sciences, Federal University of Technology, P.M.B 65, Minna, Niger State, Nigeria
²Mathematics Department, Faculty of Physical Sciences, University of Ilorin, Ilorin, Kwara State, Nigeria
*Corresponding Author: g.bolarin@futminna.edu.ng

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Abstract Finding the best way to vaccinate people against infectious disease is an important issue for health workers. In this study a compartmental two-time delay SVEIRS mathematical model with pulse vaccination and saturated incidence was formulated to examine the dynamics of infectious disease in a population. The existence of the disease free periodic solution was established and the compact form was derived. From our study, it was discovered that short pulse vaccination or long latent period or long immune period will guarantee eradication of the disease in the population. Lastly, the conditions for the incurability of the disease were examined.

Keywords SVEIRS, Incurability, Permanence, Global Attractivity, Disease-Free Periodic Solution, Immunity

1. Introduction

To control infectious diseases, vaccination is always the approach recommended by health workers and epidemiologists. Controlling and eliminating diseases has been studied by many authors, see [1] and [2]. Constant vaccination strategy or approach has been the major method proposed by many WHO workers and consultants, but Agur [3] started the study of pulse vaccination strategy as an alternative to constant vaccination strategy. Since then a lot of researchers had worked on the theoretical and practical advantages of pulse vaccination strategy, see [3-7]. The application of pulse vaccination strategy to epidemiological models was restricted to SIS and SIR models, but recently researchers have been studying more complex classifications [6] and [8-10]. Pulse vaccination is the repeated application of vaccine across age cohorts.

Meng et al [11] and Jin et al [12] studied an SIR model with some people failing to obtain immunity after first dose but gained immunity after later doses.

As we know immunity to infectious diseases after being vaccinated against them might not be life long, so in this study we assume that the latent and immunity (not permanent) period are constants. Delayed models (ordinary delay or two-time delay) with nonlinear or saturated incidence have also been proposed by some researchers, see [10-16], so a two-time delay model was replicated in this study but of SVEIRS type. Precisely, we consider a saturated incidence of the form \( \beta I/(1 + \alpha S) \) and the two-time delay (latent and immunity) for this model.

The focus of this work is to determine the conditions for which the disease will be eradicated or otherwise become incurable in a population.

2. Model Formulation

In this section we are going to formulate the model by considering vaccination as an epidemiological class.

In this regard, we use the following notations and assumptions for subsequent development: Let \( S(t) \), \( V(t) \), \( E(t) \), \( I(t) \) and \( R(t) \) denote respectively the susceptible, vaccinated, exposed, infectious and recovered partitions of the population, and \( N(t) \) be the total population. We assume that new individuals enter into the susceptible class of the population at a constant recruitment rate \( A \) and death will occur to them at the rate \( \mu S(t) \) (where \( \mu \) is the per capita natural mortality rate). With the assumption of saturated incidence \( \beta I(t)/(1 + \alpha S(t)) \) the interaction between susceptible individuals and infectious individuals can be represented by \( \beta I(t)/(1 + \alpha S(t)) \), where \( \beta \) is the adequate contact rate that can lead to infection and \( \alpha \) is the saturated incidence parameter. If \( \beta_0 \) is the adequate contact of an infective and a vaccinated individual per unit time, then the interaction of infectives and vaccinated is describe by \( \beta_0 V(t)I(t)/(1 + \alpha S(t)) \).

Death will occur to the vaccinated class, exposed class, infectious class and recovered class at the rate, \( \mu'(t) \), \( \mu E(t) \), \( \mu I(t) \) and \( \mu R(t) \) respectively and there will be an additional disease induced death of infectious individuals at the rate \( \varphi I(t) \), where \( \varphi \) is the death rate due to infection. We let \( \gamma \) be the recovery parameter of infectious
individuals and $\rho$ is the average rate for vaccinated individuals to obtain immunity and move to recovered class, $\tau$ is the average immune period of the population and $\omega$, the latent period of the disease.

Note: Parameters $A, \beta, \mu, \gamma, \omega, \tau, \alpha, \varphi, \nu, \rho$ are positive constant.

So, using the formulation above we have the following system as the SVEIRS model:

$$
\begin{align*}
\dot{S}(t) &= A - \frac{\beta S(t)I(t)}{1 + \alpha S(t)} - \mu S(t) + \rho I(t - \tau)e^{-\mu t} \\
\dot{V}(t) &= -\left(\gamma + \mu + \frac{\nu \beta I(t)}{1 + \alpha S(t)}\right)V(t) \\
E(t) &= \frac{\beta S(t)I(t)}{1 + \alpha S(t)} + \frac{\nu \beta I(t)}{1 + \alpha S(t)} - \mu E(t) \\
I(t) &= \frac{\beta S(t - \omega)I(t - \omega)e^{-\omega t}}{1 + \alpha S(t - \omega)} - \mu I(t) \\
\dot{R}(t) &= \gamma V(t) + \rho I(t) \\
-(\mu + \varphi + \rho)I(t) \\
\end{align*}
$$

$$
\begin{align*}
\hat{S}(t) &= (1 - \theta)S(t) \\
\hat{V}(t) &= V(t) + \theta S(t) \\
\hat{E}(t) &= E(t) \\
\hat{I}(t) &= I(t) \\
\hat{R}(t) &= R(t) \\
\end{align*}
$$

Now, we are going to incorporate pulse vaccination strategy into our model, this will lead us to the following impulsive system:

$$
\begin{align*}
\dot{S}(t) &= A - \frac{\beta S(t)I(t)}{1 + \alpha S(t)} - \mu S(t) + \rho I(t - \tau)e^{-\mu t} \\
\dot{V}(t) &= -\left(\gamma + \mu + \frac{\nu \beta I(t)}{1 + \alpha S(t)}\right)V(t) \\
E(t) &= \frac{\beta S(t)I(t)}{1 + \alpha S(t)} + \frac{\nu \beta I(t)}{1 + \alpha S(t)} - \mu E(t) \\
I(t) &= \frac{\beta S(t - \omega)I(t - \omega)e^{-\omega t}}{1 + \alpha S(t - \omega)} - \mu I(t) \\
\dot{R}(t) &= \gamma V(t) + \rho I(t) \\
-(\mu + \varphi + \rho)I(t) \\
\end{align*}
$$

$$
\begin{align*}
\hat{S}(t) &= (1 - \theta)S(t) \\
\hat{V}(t) &= V(t) + \theta S(t) \\
\hat{E}(t) &= E(t) \\
\hat{I}(t) &= I(t) \\
\hat{R}(t) &= R(t) \\
\end{align*}
$$

where $\theta \in [0, 1, 2, ...], T$ is the period of pulsing and

$$
N(t) = S(t) + V(t) + E(t) + I(t) + R(t) \forall t \geq 0
$$

$Z^+ = \{0, 1, 2, ...\}$.
3. Model Description and Preliminaries

By system (2) and equation (3) we have;

\[ N'(t) = A - \mu N(t) - \phi I(t) \]  
(4)

(by summing from the first to the fifth equation of (2)), therefore the total population is not constant over time. It follows from (1) that

\[ A - (\mu + \phi) N(t) \leq N'(t) = A - \mu N(t) - \phi I(t) \leq A - \mu N \]

\[ \Rightarrow \frac{A}{\mu + \phi} \leq \liminf_{t \to \infty} N(t) \leq \limsup_{t \to \infty} N(t) \leq \frac{A}{\mu} \]  
(5)

Following the approach of Gao et al [10,17] and Song et al [14], we can see from (2) that the first, second and fourth equations of the system does not contain the variables \( E(t) \) and \( R(t) \), so we are going to focus on the following equivalent system of (2).

\[
\begin{align*}
\frac{dS(t)}{dt} &= A - \beta S(t)I(t) + \mu \left[ S(t) + I(t) - \frac{S(t) + I(t)}{1 + \alpha S(t)} \right] \left[ 1 + \frac{1 + \alpha S(t)}{S(t)} \right] \nonumber \\
\frac{dV(t)}{dt} &= -\mu V(t) + \beta S(t)I(t) + \mu V(t) + \frac{\mu V(t)}{1 + \alpha S(t)} - \beta S(\tau - \tau)I(\tau - \tau) - \beta S(\tau - \tau)I(\tau - \tau) \nonumber \\
\frac{dI(t)}{dt} &= \beta S(t)I(t) - \mu I(t) + \beta S(\tau - \tau)I(\tau - \tau) - \beta S(\tau - \tau)I(\tau - \tau) \nonumber \\
+ \left[ \frac{\mu V(t)}{1 + \alpha S(t)} - \beta S(\tau - \tau)I(\tau - \tau) - \beta S(\tau - \tau)I(\tau - \tau) \right] \epsilon^{(\mu + \rho)\tau} \epsilon^{-\rho \tau} - \beta S(t - \tau)I(t - \tau) - \beta S(t - \tau)I(t - \tau) \\
\frac{dN(t)}{dt} &= A - \mu N(t) - \phi I(t) \\
S(t') &= (1 - \theta)S(t'), \quad V(t') = V(t') + \Theta S(t'), \quad I(t') = I(t'), \quad N(t') = N(t')
\end{align*}
\]

(6)

\[ t = kT, k \in \mathbb{Z}^+ \]

The initial conditions for (6) are of the form

\[
\begin{align*}
S(\xi) &= \phi_1(\xi), \quad V(\xi) = \phi_2(\xi), \quad I(\xi) = \phi_3(\xi), \quad N(\xi) = \phi_4(\xi); \quad -l \leq \xi \leq 0
\end{align*}
\]

(7)

We have,

\[ \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \]  
such that \( C = C([-l, 0], \mathbb{R}_+^4) \) and \( C_1 \) is the space of continuous functions on \([-l, 0]\] with uniform norm; where

\[ l = \max \{\tau, \omega\} \]  
moreso, \( \mathbb{R}_+ = [0, \infty) \);

\[ \mathbb{R}_+^4 = \{J \in \mathbb{R}^4 : J \geq 0\} \]

The solution of (6) is a piecewise continuous function \( J : \mathbb{R}_+ \to \mathbb{R}_+^4, J(t) \) has a point of discontinuity at \(-\kappa T (\kappa \in J^+)\) of the first kind and continuous on \((kT, (k + 1)T), k \in J^+\) and \( J(kT^-) = \lim_{t \to \kappa T} J(t) \) exists.

Since \( \phi \) is smooth then the solution of (6) exists and is unique (using definition of smooth function).

We assume that \( \phi_j(0) > 0 \forall j = 1, 2, 3, 4 \) and since \( S'(t) \) \( \to 0 \) and \( I(t) = 0 \) where ever \( I(t) = 0 \forall t \neq \kappa T, \kappa \in \mathbb{Z}^+ \).

Also, using the fact that

\[
S(kT^+) = (1 - \theta)S(kT^-), V(kT^+) = V(kT^-) + \theta S(kT^-),
\]

\[ I(kT^+) = I(kT^-), N(kT^+) = N(kT^-) \]

for \( t \neq kT, \kappa \in \mathbb{Z}^+ \).

Now from (6), since \( N(t) = S(t) + V(t) + I(t) \) then we have the following lemma.

**Lemma 1**

Suppose \( J(t) \) is a solution of (6) with initial condition (7) then,

\[ S(t) \leq \frac{A}{\mu}, \quad V(t) \leq \frac{A}{\mu}, \quad I \leq \frac{A}{\mu} \forall t \text{ large enough.} \]

**Proof**

Since \( N(t^+) = N(t^+) = N(t^+) \forall t \geq 0 \) then \( N(t) \) is continuous on \( t \in [0, +\infty) \) and by (4)

\[
\begin{align*}
\frac{dS(t)}{dt} &= A - \mu N(t) - \phi I(t), \quad \text{We have}
\frac{dN(t)}{dt} &= A - \mu N(t) - \phi I(t) - A - \mu N(t)
\end{align*}
\]

\[ \Rightarrow \liminf_{t \to \infty} N(t) \leq \limsup_{t \to \infty} N(t) \leq \frac{A}{\mu}. \]

So \( N(t) \) is uniformly bounded. If there exist a positive integer \( k \), by definition of \( N(t) \) we have that

\[ S(t) \leq \frac{A}{\mu}, \quad V(t) \leq \frac{A}{\mu}, \quad I(t) \leq \frac{A}{\mu} \forall t \geq kT \]

More so, biologically we assume that \( \phi_j(0) > 0 \forall j = 1, 2, 3, 4 \) then the system (6) is positively invariant in the closed set:

\[ \Theta = \{(S, V, I, N) \in \mathbb{R}_+^4 : S + V + I \leq \frac{A}{\mu}, \quad N \leq \frac{A}{\mu}\}. \]

**Definition 1 Uniform Persistence**

System (6) is said to be uniformly persistent if there is a \( \lambda > 0 \)

(independent of the initial conditions) such that every solution \( (S(t), V(t), I(t), N(t)) \) of system (6) with initial conditions (7) satisfies
Definition 2 Permanence
System (6) is said to be permanent in $\Theta$ if there exists a compact region $\Theta_0 \in \text{int } \Theta$ such that every solution of system (6) with initial conditions (7) will eventually enter and remain in region $\Theta_0$.

Lemma 2 [10]
Consider the following impulsive system
\[
\begin{align*}
  u(t) &= a - bu(t); \\
  u(t^+) &= (1 - \theta)u(t^-); \\
  t &\neq \kappa T
\end{align*}
\]  
where $a > 0, b > 0, 0 < \theta < 1$. Then there exists a unique positive periodic solution of system (8) given by
\[
\tilde{u}_{a,b}(t) = \frac{a(1 - \theta)(1 - e^{-b(T - \kappa T)})}{b(1 - \theta)e^{-bT}}
\]  
which is asymptotically stable, where
\[
\tilde{u}_{a,b} = a(1 - \theta)(1 - e^{-b(T - \kappa T)})
\]

Lemma 3 [18]
Let us consider the following impulsive differential inequalities:
\[
\begin{align*}
  w'(t) &\leq (\geq)P(t)w(t) + g(t); \\
  w(t^+) &\leq (\geq)\delta w(t^-) + b_k; \\
  t &\neq t_k, k \in N
\end{align*}
\]  
where $P(t), g(t) \in C[R_+], \delta > 0$ and $b_k$ are constants.
Assume, 
\begin{enumerate}
  \item the sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \ldots$, with $\lim_{t \to \infty} t_k = \infty$;
  \item $w \in PC^1[R_+, R]$ and $w(t)$ is left continuous at $t_k, k \in N$. Then
\end{enumerate}
\[
\begin{align*}
  w(t) &\leq (\geq)w(t_0) \prod_{k \in I} d_k \exp \left[ \sum_{s \in \pi(I)} \prod_{i \in s} \int_{t_{i-1}}^{t_i} \left( \int_{s} P(s) \, ds \right) \right] \prod_{k \in I} d_k \exp \left[ \int_{t_0}^{t_k} \left( \int_{s} P(s) \, ds \right) \right] b_k;
\end{align*}
\]
\[
\begin{align*}
  w(t) &\leq (\geq)w(t_0) \prod_{k \in I} d_k \exp \left[ \sum_{s \in \pi(I)} \prod_{i \in s} \int_{t_{i-1}}^{t_i} \left( \int_{s} P(s) \, ds \right) \right] b_k;
\end{align*}
\]

Lemma 4 [9] and [19]
Consider the following equation,
\[
\dot{x}(t) = ax(t - \omega) - a_2 x(t)
\]
where $a, a_2, \omega > 0; x(t) > 0$ for $-\omega \leq t \leq 0$. We have;
\begin{enumerate}
  \item If $a < a_2$; then $\lim_{t \to \infty} x(t) = 0$
  \item If $a > a_2$; then $\lim_{t \to \infty} x(t) = +\infty$
\end{enumerate}

3.1. Model Analysis
3.1.1. Global Attractivity of Disease-free Periodic Solution
From (5) and the fact that
\[
\frac{dS(t)}{dt} = A - \beta S(t)I(t) - \mu S(t) + \mu I(t - \tau) e^{-\mu \tau}
\]
we have
\[
\frac{dS(t)}{dt} \geq A - \left( \frac{\mu + A\beta}{\mu + \alpha A} \right) S(t)
\]

Now we show that there exist a disease-free periodic solution for system (6), that is the infectious individuals are absent from the population (i.e. $I(t) = 0 \forall t \geq 0$).
So if $I(t) = 0 \forall t \geq 0$ then (6) simplifies to
\[
\begin{align*}
  \frac{dS(t)}{dt} &= A - \mu S(t) \\
  \frac{dV(t)}{dt} &= -\left( \gamma + \mu \right) V(t) \\
  \frac{dN(t)}{dt} &= A - \mu N(t) \\
  S(t^+) &= (1 - \theta)S(t^-) \\
  V(t^+) &= V(t^-) + \theta S(t^-) \\
  N(t^+) &= N(t^-)
\end{align*}
\]
which is the disease-free system whose unique solution exists [9,20] and derived subsequently.
For $I(t) = 0 \forall t \geq 0$ we have
\[
\lim_{t \to \infty} N(t) = \frac{A}{\mu}
\]
Therefore using the fact that $N(t) = S(t) + V(t) + I(t)$ we have the following equations
\[
\begin{align*}
  V(t) &= \frac{A}{\mu} S(t)
\end{align*}
\]
and
\[
\begin{align*}
  \frac{dS(t)}{dt} &= A - \mu S(t) \\
  \frac{dV(t)}{dt} &= -\left( \gamma + \mu \right) V(t) \\
  S(t^+) &= (1 - \theta)S(t^-) \\
  V(t^+) &= V(t^-) + \theta S(t^-) \\
  N(t^+) &= N(t^-)
\end{align*}
\]  
Solving (11) between pulses and using the discrete dynamical system determined by the stroboscopic map and the integral of the system, we have
\[
\begin{align*}
  \dot{S}_{t,\mu}(t) &= A + \left( S'_{t,\mu} - \frac{A}{\mu} e^{\mu(t - \kappa T)} \right) \gamma T < t \leq (\kappa + 1)T \\
  S'_{t,\mu} &= \frac{A}{\mu} \left[ 1 - \theta e^{-\mu(t - \kappa T)} \right] \gamma T < t \leq (\kappa + 1)T, \kappa \in \mathbb{Z}^+
\end{align*}
\]
By (10) and (12) we have

\[ V^*(t) = \frac{A}{\mu} \left[ \frac{\theta (1 - e^{-\mu T}) e^{-(\mu + \gamma)(t-kT)}}{(1-e^{-(\mu + \gamma)T})(1-(1-\theta)e^{-\mu T})} \right] \]  

(13),

with \( kT < t \leq (k+1)T \); \( k \in \mathbb{Z}^+ \) and

\[ S^*(0^+) = S^*(kT^+) \]

\[ = \frac{A}{\mu} \left[ \frac{1 - \theta}{1-(1-\theta)e^{-\mu T}} \right] \]

\[ V^*(0^+) = V(kT^+) \]

\[ = \frac{A}{\mu} \left[ \frac{1 - \theta}{1-(1-\theta)e^{-\mu T}} \right] \]

\( ; t = kT \)

By Lemma 2, (13) is GAS and the solution of (9) is

\[ S(t) = \frac{A}{\mu} + \left[ \frac{A}{\mu} \left( \frac{1 - \theta}{1-(1-\theta)e^{-\mu T}} \right) - \frac{A}{\mu} \right] e^{-(\mu + \gamma)(t-kT)} \]

\[ = S(0^+) - \frac{A}{\mu} \left( \frac{1 - \theta}{1-(1-\theta)e^{-\mu T}} \right) \]

\[ V(t) = V(0^+) e^{-\mu T} \]

\( \kappa T < t \leq (k+1)T ; \quad \kappa \in \mathbb{Z}^+ \)

So, the disease-free periodic solution of (6) is;

\[ (S^*(t), \frac{A}{\mu} - S^*(t), 0, \frac{A}{\mu}) \]

Next we determine the global attractivity condition of the disease-free periodic solution \( (S^*(t), (A/\mu) - S^*(t), 0, A/\mu) \) of system (6).

First we consider the following lemma;

**Lemma 5**

System (9) has a unique positive solution

\( (S^*(t), V^*(t), \frac{A}{\mu}) \),

meaning that the system (6) has a disease-free periodic solution

\( (S^*(t), V^*(t), 0, \frac{A}{\mu}) \) \forall \ t \in (kT, (k+1)T] \text{, \( k \in \mathbb{Z}^+ \)},

so for any solution \( (S(t), V(t), I(t), N(t)) \) of system (6), then \( S(t) \rightarrow S^*(t) \), \( V(t) \rightarrow V^*(t) \) and

\[ N(t) \rightarrow \frac{A}{\mu} \text{ as } t \rightarrow \infty. \]

**Proof**

The proof follows from Lemma 2.

Now, we set the basic reproduction number \( R^* \) to

\[ R^* = \frac{A \beta e^{-\mu T}}{\mu + \phi} \left( \frac{r_1 r_2}{\mu (r_1 + \theta + \alpha r_2)} + \frac{\theta r_1 r_3}{\mu (r_2 + \theta)(1-r_2) + a \theta r_1 r_3} \right) \]

where \( r_1 = \mu + pe^{-\mu T}, r_2 = e^{\mu T} - 1, r_3 = e^{(\mu + \gamma)T} \)

then we state and prove the following theorem.

**Theorem 1**

If \( R^* < 1 \), then the disease-free periodic solution \( (S^*(t), V^*(t), 0, \frac{A}{\mu}) \) of system (6) is globally attractive.

**Proof**

Let \( (S(t), V(t), I(t), N(t)) \) be a solution of system (6) with initial conditions (7). Since \( R^* < 1 \), we can choose an \( \epsilon > 0 \) sufficiently small such that;

\[ (A r_2 \mu (r_1 + \theta + \alpha r_2)) + (\frac{A \theta r_1 r_3}{\mu (r_2 + \theta)(1-r_2) + a \theta r_1 r_3}) > (\mu + \phi) < 0 \]

\[ \Rightarrow (\mu + \phi) \left( \frac{A r_2 \mu (r_1 + \theta + \alpha r_2)}{\mu (r_2 + \theta)(1-r_2) + a \theta r_1 r_3} + \epsilon \right) < (\mu + \phi) \]

From the first and fifth equations of system (6) it follows that for \( \kappa \geq \kappa_1 \);

\[ S(t) \leq A - \mu S(t) + \rho e^{-\mu T} ; \quad t \neq \kappa T , \quad \kappa \in \mathbb{Z}^+ \]

\[ S(t) = (1-\theta) S(t) ; \quad t = \kappa T , \quad \kappa \in \mathbb{Z}^+ \]

By Lemma 3, we have

\[ S(t) \leq S(\kappa T^+) \prod_{\kappa_1 < \kappa < \kappa} (1-\theta) \exp \int_{\kappa T}^{\kappa T} -\mu ds + \int_{\kappa T}^{\kappa T} \prod_{\kappa < \kappa < \kappa} (1-\theta) \exp \int_{\kappa T}^{\kappa T} -\mu ds (A + \rho e^{-\mu T}) ds \]

\[ = S_1 + S_2 \]

where

\[ S_1 = S(\kappa T^+) \prod_{\kappa_1 < \kappa < \kappa} (1-\theta) \exp \int_{\kappa T}^{\kappa T} -\mu (T T') \]

\[ = S(\kappa T^+)(1-\theta)^{1/\gamma} e^{-\mu T} \]

and

\[ S_2 = \frac{A + \rho e^{-\mu T}}{\mu} \left[ \frac{(1-\theta)(e^{\mu T} - 1)e^{\alpha T} + \theta r_1 r_3}{1-\theta - \rho e^{-\mu T}} - e^{\rho T(1/\gamma)} \right] \]
so if we set

\[
q(t) = e^{-\mu t} \left[ S(\kappa_t T^+) (1 - \theta)^{1/\theta} e^{\kappa_t \mu T} - \frac{A + \rho e^{-\mu t} (1 - \theta)^{1/\theta} e^{\kappa_t \mu T}}{\mu} \right] e^{\mu t T - 1 + \theta}
\]

\[
q(t) \leq e^{-\mu t} S(\kappa_t T^+) (1 - \theta)^{1/\theta} e^{\kappa_t \mu T} = e^{-\mu t} S(\kappa_t T^+) (1 - \theta)^{1/\theta} e^{\kappa_t \mu T} < e^{-\mu t} S(\kappa_t T^+) e^{\kappa_t \mu T}
\]

and

\[
S(t) \leq S_1 + S_2 = q(t) + \frac{A + \rho e^{-\mu t}}{\mu} \left[ 1 - \frac{\theta}{e^{\mu t - 1 + \theta}} \right] + \varepsilon = \Psi_1
\]

From these, we have

\[
S(t) \leq e^{-\mu t} S(\kappa_t T^+) e^{\kappa_t \mu T} + \frac{A + \rho e^{-\mu t}}{\mu} \left[ 1 - \frac{\theta}{e^{\mu t - 1 + \theta}} \right] \Rightarrow \limsup_{t \to \infty} S(t) \leq \frac{A + \rho e^{-\mu t}}{\mu} \left[ 1 - \frac{\theta}{e^{\mu t - 1 + \theta}} \right]
\]

So if there exist a positive integer \( \kappa_2 \geq \kappa_1 \) and an arbitrarily small positive constant \( \varepsilon \) such that \( t \geq \kappa_2 T \), we have

\[
S(t) \leq \frac{A + \rho e^{-\mu t}}{\mu} \left[ 1 - \frac{\theta}{e^{\mu t - 1 + \theta}} \right] + \varepsilon = \Psi_1
\]

Again from the second and sixth equations of (6), we have

\[
\forall \kappa \geq \kappa_1
\]

\[
V'(t) \leq -\gamma + \mu V(t), \quad t \neq \kappa T, \quad \kappa \in \mathbb{Z}^+
\]

\[
V(t) = V(0) + \theta S(t) \quad t = \kappa T, \quad \kappa \in \mathbb{Z}^+
\]

We consider the following comparison impulsive differential system

\[
y'(t) = -(\gamma + \mu)y; \quad t \neq \kappa T \quad \kappa \in \mathbb{Z}^+
\]

\[
y(t) = y + \theta S; \quad t = \kappa T \quad \kappa \in \mathbb{Z}^+
\]

Solving the equation between pulses and using discrete dynamical system approach, we have

\[
y'(t) = \frac{\theta e^{(\gamma + \mu) t} (A + \rho e^{-\mu t})}{\mu (1 - e^{(\gamma + \mu) t})} \left[ 1 - \frac{\theta}{e^{\mu t - 1 + \theta}} \right]; \quad kT < t \leq (k+1)T \quad \kappa \in \mathbb{Z}^+
\]

\[
y'(0) = \frac{\theta (A + \rho e^{-\mu t})}{\mu (1 - e^{(\gamma + \mu) t})} \left[ 1 - \frac{\theta}{e^{\mu t - 1 + \theta}} \right]; \quad t = \kappa T, \quad \kappa \in \mathbb{Z}^+
\]

which is GAS by Lemma 2.

Let \( S(t), V(t), I(t), N(t) \) be the solution of system (6) with initial condition (7) and

\[
S(0^+) = S_0 > 0, \quad V(0^+) = V_0 > 0, \quad y(t) \quad \text{be the solution of system (16) with initial condition} \quad y(0^+) = V_0.
\]

By comparison theorem for impulsive differential equation [20], there exists an integer \( \kappa > 0 \) such that

\[
V(t) < y(t) < y'(t) + \varepsilon_0; \quad \kappa T < t \leq (\kappa + 1)T, \quad \kappa \in \mathbb{Z}^+
\]

\[
V(t) < y' + \varepsilon_0 + \frac{(A + \rho e^{-\mu t})}{\mu (1 - e^{(\gamma + \mu) t})} \left[ 1 - \frac{\theta}{e^{\mu t - 1 + \theta}} \right] + \varepsilon_1 = \Psi_2
\]

From (15), (17) and third equation of (6), \( \forall t > \kappa_2 T + \omega \) we have,

\[
I(t) \leq \left( \frac{\beta e^{-\mu \theta} \Psi_1 I(t - \omega)}{1 + \alpha \Psi_1} \right) + \left( \frac{\nu \beta e^{-\mu \theta} \Psi_2 x(t - \omega)}{1 + \alpha \Psi_1} \right) - (\mu + \varphi + \rho) I(t)
\]

Let us consider the following impulsive equation as the comparison equation

\[
x'(t) = \left( \frac{\beta e^{-\mu \theta} \Psi_1 x(t - \omega)}{1 + \alpha \Psi_1} \right) + \left( \frac{\beta e^{-\mu \theta} \Psi_2 x(t - \omega)}{1 + \alpha \Psi_1} \right) - (\mu + \varphi + \rho) x(t)
\]

And so by Lemma 5 we have,

\[
\lim_{t \to \infty} x(t) = 0
\]

Since \( I(s) = x(s) = \phi > 0 \quad \forall \quad s \in [-l, 0] \), then by the comparison theorem of differential equations and the fact that the solution of system (6) are non-negative with \( I(t) \geq 0 \) then we have \( I(t) \to 0 \) as \( t \to \infty \).

Next we assume that \( 0 < I(t) < \varepsilon \quad \forall \quad t \geq 0 \). From (6) we have

\[
S'(t) \geq A - \mu S - \beta \varepsilon
\]

\[
V'(t) \geq -\nu \beta \varepsilon - (\gamma + \mu) V(t)
\]

If we have

\[
\begin{align*}
\bar{b}(t) & = A - \mu b - \beta \varepsilon \\
\bar{c}(t) & = -\nu \beta \varepsilon - (\gamma + \mu) c(t)
\end{align*}
\]

\[
\begin{align*}
\bar{b}(t') & = (1 - \theta) \bar{b} \\
\bar{c}(t') & = \bar{c} + \theta \bar{b}
\end{align*}
\]

\[
\begin{align*}
\bar{b}(0^+) & = S(0^+) \\
\bar{c}(0^+) & = V(0^+)
\end{align*}
\]
and assuming that $h(t) \to S^*(t)$, $c(t) \to V^*(t)$ and 

$$N \to \frac{A}{\mu} \quad \text{as} \quad \varepsilon \to 0$$

where $(\tilde{h}(t), \tilde{c}(t))$ is a unique positive periodic solution of

(19).

From (25) we have

$$\tilde{b}_1 = \frac{A - \beta \varepsilon}{\mu} \left(1 - \frac{\theta e^{-\mu(t-kT)}}{1-(1-\theta)e^{-\mu \kappa T}}\right)$$

$$\tilde{c}_1 = -\frac{\nu \beta \varepsilon}{\mu + \gamma}$$

$$+ \frac{A - \beta \varepsilon}{\mu} \left(1 - \frac{\theta e^{-\mu \kappa T}}{1-(1-\theta)e^{-\mu \kappa T}}\right) e^{-(\mu+\gamma)(t-kT)}$$

$$\forall \kappa T < t < (\kappa+1)T$$

Again, using the comparison theorem of impulsive system [11], we have the following:

If there exist $\varepsilon_i > 0 \quad \forall T_i > 0$ then

$$\left\{\begin{array}{l}
S(t) > \tilde{b}_1 - \varepsilon_1 \\
V(t) > \tilde{c}_1 - \varepsilon_1
\end{array}\right. \forall t > T$$

(20).

From (6) we have

$$S'(t) \leq A - \mu S(t) + \rho \varepsilon e^{-\mu t}$$

$$V'(t) \leq -\gamma V(t) - \mu V(t)$$

So, if we have

$$\tilde{b}_2, \tilde{c}_2$$ as the unique positive periodic solution of

$$\begin{cases}
\tilde{b}_2(t) = A - \mu \tilde{b}_2(t) - \rho \varepsilon e^{-\mu t} \\
\tilde{c}_2(t) = -(\gamma + \mu)\tilde{c}_2(t)
\end{cases} \quad t \neq kT; \quad \kappa \in \mathbb{Z}^+
\begin{cases}
\tilde{b}_2(t^+) = (1-\theta)\tilde{b}_2(t) \\
\tilde{c}_2(t^+) = \tilde{c}_2 + \theta \tilde{b}_2
\end{cases} \quad t = kT; \quad \kappa \in \mathbb{Z}^+
\begin{cases}
\tilde{b}_2(0^+) = S(0^+) \\
\tilde{c}_2(0^+) = V(0^+)
\end{cases}
$$

Then

$$S(t) \leq \tilde{b}_2(t), V(t) \leq \tilde{c}_2(t),$$

and $\tilde{b}_2(t) \to S^*(t)$, $\tilde{c}_2(t) \to V^*(t)$ as $\varepsilon \to 0$.

So, from (21) we have

$$\tilde{b}_2 = A + \rho \varepsilon e^{-\mu t} \left(1 - \frac{\theta e^{-\mu(t-kT)}}{1-(1-\theta)e^{-\mu \kappa T}}\right)$$

$$\tilde{c}_2 = \theta e^{-\mu \kappa T} \left(1 - \frac{\theta e^{-\mu \kappa T}}{1-(1-\theta)e^{-\mu \kappa T}}\right)$$

$$\forall \kappa T < t < (\kappa+1)T$$

Now, using the comparison theorem of impulsive system, as presented in [21], we have the following; for $\varepsilon_i > 0 \exists T_i > 0$ such that

$$\left\{\begin{array}{l}
S(t) < \tilde{b}_2(t) + \varepsilon_i \\
V(t) < \tilde{c}_2(t) + \varepsilon_i
\end{array}\right. \forall t > T$$

(22).

If we let $\varepsilon \to 0$, then from (20) and (22) we have

$$\left\{\begin{array}{l}
S'(t) - \varepsilon_i < S(t) < S^*(t) + \varepsilon_i \\
V'(t) - \varepsilon_i < V(t) < V^*(t) + \varepsilon_i
\end{array}\right.$$

as $t \to \infty$

$$\Rightarrow S(t) \to S^*(t), V(t) \to V^*(t) \quad \text{as} \quad t \to \infty.$$

From the above and using the fact that

$$\lim_{t \to \infty} N(t) = \frac{A}{\mu},$$

the proof is complete.

Next, let

$$\tau_s = \frac{1}{\mu} \ln \left[1 + \frac{\theta(\mu + \varphi + \rho)}{A \beta e^{-\mu \tau} - \mu(\mu + \varphi + \rho)}\right]$$

$$\theta_s = \frac{(e^{-\mu \tau} - 1)(A \beta e^{-\mu \tau} - \mu(\mu + \varphi + \rho))}{\mu(\mu + \varphi + \rho)}$$

and

$$\omega_s = -\frac{1}{\mu} \ln \frac{\mu(\mu + \varphi + \rho)(1-(1-\theta)e^{-\mu \tau})}{A \beta (1-e^{-\mu \tau})}$$

**Corollary 1**

If $\theta > \theta_s$ or $\omega > \omega_s$, the infection-free periodic solution

$$\left\{\begin{array}{l}
S(t), \frac{A}{\mu} - S^*(t), 0, \frac{A}{\mu}
\end{array}\right.\right.$$ is globally attractive.

**Corollary 2**

For system (6) with initial condition (7) we have the infection-free periodic solution

$$\left\{\begin{array}{l}
S(t), \frac{A}{\mu} - S^*(t), 0, \frac{A}{\mu}
\end{array}\right.\right.$$ is globally attractive provided that;

(i) $A \beta e^{-\mu \tau} > \mu(\mu + \varphi + \rho)$ and $\tau < \tau_s$

(ii) $A \beta (1-e^{-\mu \tau}) > \mu(\mu + \varphi + \rho)(1-(1-\theta)e^{-\mu \tau})$}

and $\omega > \omega_s$. 

---
3.1.2. Incurability or Permanence of the Disease

The system (6) is said to be permanent if there are positive constants \( q_i, Q_i, i = 1, 2, 3, 4 \) and a finite time \( T_0 \) such that for all solution \( (S(t), V(t), I(t), N(t)) \) with initial values,

\[
S(0^+) > 0, V(0^+) > 0, I(0^+) > 0 \text{ and } N(0^+) > 0; \quad q_1 \leq S(t) \leq Q_1, q_2 \leq V(t) \leq Q_2, q_3 \leq I(t) \leq Q_3; \quad q_4 \leq N(t) \leq Q_4 \quad \forall \ t \geq T_0
\]

Let,

\[
R_e = \frac{A \beta e^{-\mu_0}}{(1 + \alpha)(\mu + \varphi + \rho)} \left( \frac{1 - e^{-(\mu + \varphi + \rho)t}}{1 - (1 - \theta)e^{-\mu t}} \right) \left( \frac{1 - e^{-(\mu + \varphi + \rho)t}}{1 - (1 - \theta)e^{-\mu t}} \right)
\]

and

\[
I_* = \frac{\mu}{\beta} (R_e - 1)
\]

**Theorem 2**

If \( R_e > 1 \), then the disease is permanently incurable, i.e. there exist a positive constant \( q_2 \) such that \( I(t) \geq q_2 \) for a large enough \( t \).

**Proof**

Suppose \( N(t) = (S(t), V(t), I(t), N(t)) \) is any nonnegative solution of (6) with initial conditions (7) and so from the third equation of (7), we have the following equivalent equation:

\[
I'(t) = \frac{\beta e^{-\mu_0} S(t)I(t)}{1 + \alpha S(t)} + \frac{v \beta e^{-\mu_0} V(t)I(t)}{1 + \alpha S(t)} + \beta e^{-(\omega + \tau)\mu} e^{-\varphi t} \left[ S(t)I(t) - \frac{S(t - \omega - \tau)I(t - \omega - \tau)}{1 + \alpha S(t)} \right] - \beta e^{-(\omega + \tau)\mu} e^{-\varphi t} S(t)I(t) - (\mu + \varphi + \rho)I(t)
\]

\[
= \frac{\beta e^{-\mu_0} S(t)I(t) + v \beta e^{-\mu_0} V(t)I(t)}{1 + \alpha S(t)} - \beta e^{-(\omega + \tau)\mu} e^{-\varphi t} S(t)I(t) - (\mu + \varphi + \rho)I(t)
\]

\[
= \frac{\beta e^{-\mu_0} S(t)I(t) + v \beta e^{-\mu_0} V(t)I(t)}{1 + \alpha S(t)} - \beta e^{-(\omega + \tau)\mu} e^{-\varphi t} S(t)I(t) - (\mu + \varphi + \rho)I(t)
\]

From (23), we calculate the derivative of \( R(t) \) evaluated along the solution of (6) as

\[
D(t) = \frac{\beta e^{-\mu_0} S(t)I(t) + v \beta e^{-\mu_0} V(t)I(t)}{1 + \alpha S(t)} + \frac{v \beta e^{-\mu_0} V(t)I(t)}{1 + \alpha S(t)} - \beta e^{-(\omega + \tau)\mu} e^{-\varphi t} S(t)I(t) - (\mu + \varphi + \rho)I(t)
\]

\[
D(t) = \frac{\beta e^{-\mu_0} S(t)I(t) + v \beta e^{-\mu_0} V(t)I(t)}{1 + \alpha S(t)} + \frac{v \beta e^{-\mu_0} V(t)I(t)}{1 + \alpha S(t)} - \beta e^{-(\omega + \tau)\mu} e^{-\varphi t} S(t)I(t) - (\mu + \varphi + \rho)I(t)
\]

(24).
Since $R_0 > 1$, we can easily see that $I_0 > 0$ and there exist a positive constant $2 \varepsilon_i$ small enough such that;

$$\frac{\beta e^{-\mu \phi}}{(\mu + \varphi + \rho)} \left( \frac{A_i (1 - e^{-(\mu + \varphi + \rho) t})}{1 + \alpha A_i} \right) + \nu A_2 > 1 \quad (25),$$

where

$$A_i = \frac{A}{\beta \lambda + \mu} \left( \frac{1 - (1 - \theta) e^{-((\beta \lambda + \mu) \tau) t}}{1 - (1 - \theta) e^{-(\beta \lambda + \mu) \rho) t}} - \varepsilon_i \right)$$

$$A_2 = \frac{A}{\beta \lambda + \mu} \left( \frac{\theta e^{-(\beta \lambda + \mu + \rho) \tau) t}}{1 - (1 - \theta) e^{-(\beta \lambda + \mu) \rho) t}} - \varepsilon_i \right).$$

Now, we assume that for any $t_0 > 0$ it is not possible that $I(t) < I_0$ for $t \geq t_0$. By contradiction, suppose that this is not valid, then there exist $t_0 > 0$ such that $I(t) < I_0$ for $t \geq t_0$.

From system (6) we have

$$S(t) \geq A - \beta S(t) I_0 - \mu S(t)$$

$$V(t) \geq -(\gamma + \mu) V(t) - \nu V(t) I_0$$

$$S(t) = (1 - \theta) S(t)$$

$$V(t) = V(t) + \Theta S(t)$$

Consider the following comparison impulsive system for $t \geq t_0$:

$$t \neq \kappa T; \kappa \in \mathbb{Z}^+$$

$$q'(t) = A - (\mu + \beta I) q(t)$$

$$q(t) = (1 - \theta) q(t)$$

By Lemma 2, we have

$$q(t) = \frac{A}{\mu + \beta I} + \left( q^* - \frac{A}{\mu + \beta I} \right) e^{-(\mu + \beta I) (t - \kappa T)}$$

as the unique positive periodic solution of (26) which is globally asymptotically stable with

$$q^* = \frac{A}{\mu + \beta I} \left( \frac{1 - (1 - \theta) e^{-(\mu + \beta I) \tau) t}}{1 - (1 - \theta) e^{-(\mu + \beta I) \rho) t}} \right)$$

By comparison theorem of impulsive differential equation, there is $T_1 \geq (t_0 + \omega + \tau)$ such that

$$S(t) > \frac{A}{\beta \lambda + \mu} \left( \frac{1 - (1 - \theta) e^{-(\beta \lambda + \mu) \tau) t}}{1 - (1 - \theta) e^{-(\beta \lambda + \mu) \rho) t}} \right) - \varepsilon_i = A_1$$

$$V(t) > \frac{A}{\beta \lambda + \mu} \left( \frac{\theta e^{-(\beta \lambda + \mu + \rho) \tau) t}}{1 - (1 - \theta) e^{-(\beta \lambda + \mu) \rho) t}} - \varepsilon_i = A_2$$

$\Rightarrow$

$$S(t) > A_1$$

$$V(t) > A_2$$

From (24) and (27) we have that

$$R(t) > (\mu + \varphi) I(t) \left[ \beta e^{-\mu \phi} \left( \frac{1 - e^{-(\mu + \varphi) t}}{\mu + \varphi (1 + \alpha)} \right) A_i - \frac{\nu A_2}{(\mu + \varphi) (1 + \alpha)} \right]$$

for $t \geq t_i$$

$$t \neq \kappa T; \kappa \in \mathbb{Z}^+$$

Set

$$I_i = \min_{t \leq t_i} I(t)$$

We want to show that $I(t) \geq I_i$ for $t \geq t_i$.

Suppose the contrary, there is a $t_2 \geq t_1 + \omega + \tau$ such that $I(t_2) = I_i$ and $I(t) \geq I_i$ for $t \in [t_1, t_2]$.

From (6), (25) and (28) we see that

$$I(t_i) \geq \frac{\beta e^{-\mu \phi} (1 - e^{-(\mu + \varphi) \tau) t}}{\mu + \varphi (1 + \alpha)} A_i - \frac{\nu A_2}{(\mu + \varphi) (1 + \alpha)}$$

which is a contradiction. Hence the claim is true.

From (28), we have that

$$R(t) > (\mu + \varphi) I(t) (R_0 - 1) > 0$$

$\Rightarrow$ $R(t) \rightarrow \infty$ as $t \rightarrow +\infty$

This contradicts

$$R(t) \leq \frac{A}{\mu + \beta e^{-\mu \phi} \left( 1 - e^{-(\mu + \varphi) \tau) t} + \nu \right)}$$

Also for $t$ large enough we have two other cases to examine:

(i) $I(t) \geq I_i$

(ii) $I(t)$ oscillate about $I_i$ for $t$ large enough

$$m = \min \left\{ \frac{I_i}{2}, m_2 \right\}$$

Set

We hope to show that $I(t) \geq m_2$. For the first case ($I(t) \geq I_i$) the conclusion is obvious.

For the second case, if there exist $t^*$ (large enough) such that

$$I(t^*) = I_i, S(t) > A_i \text{ and } V(t) > A_2 \text{ for } t \geq t^*$$

$I(t)$ is uniformly continuous because it is not affected by impulse and it is bounded.

Therefore there exist a

$\theta(0 < \theta < \tau)$ which is independent of the choice of $t^*$.
such that \( I(t) > \frac{I^*}{2} \forall t^* \leq t \leq t^* \).

From the condition that

\[
\frac{A_1 e^{-\mu(t+\rho)} + \nu A_2 e^{-\mu(t+\tau)}}{1+\alpha} > 1
\]

then there exist \( \varepsilon > 0 \) sufficiently small such that

\[
\frac{(A_1^* - \varepsilon)e^{-\mu(t+\rho)} + \nu (A_2^* - \varepsilon)e^{-\mu(t+\tau)}}{1+\alpha} > 1
\]

\[
\Rightarrow \frac{A_1 e^{-\mu(t+\rho)} + \nu A_2 e^{-\mu(t+\tau)}}{1+\alpha} > 1
\]

From (6), (29) and (30) we have

\[
0 < t^* \leq \frac{1}{2}\left( \frac{\mu}{\nu} + \frac{\mu}{\nu} \right)
\]

\[
\Rightarrow I(t) \geq \frac{\mu}{\nu} I_0 \geq I^* \
\]

\[
\Rightarrow m_2 \leq I(t) \leq m^*_2
\]

Hence the proof of Theorem 2.

4. Conclusions

Theorem 1 states the global attractiveness of periodic solution

\[
\left( S^*(t), \frac{A}{\mu} - S^*(t), 0, \frac{A}{\mu} \right)
\]

of system (6) with initial condition (7) in domain \( \Theta \) for the case \( R^* < 1 \), i.e. the disease will be eradicated from the population. Corollary 1 and corollary 2 implied that the disease will be eradicated if the pulse vaccination rate is larger than \( \theta_e \), the latent period is longer than \( \omega_e \) and the infectivity period is shorter than \( \tau_e \). In this work, we have studied the delayed SVEIRS epidemic model with pulse vaccination and saturated incidence. We have been able to determine the conditions for which the disease will be eradicated in the population through the use of pulse vaccination strategy. Also we found that a smaller pulse vaccination rate or a shorter latent period of the disease or a shorter immunity period could lead to the disease been permanent in the population. Furthermore, the inclusion of saturated incidence in term of susceptible, allowed us to study the dynamics of the diseases with respect to population with very high proportion of it prone to being infected.

Lastly, with this new model, optimal control theory can be employed to determine the cost benefit of maximizing the recovered individuals and minimizing the infected and susceptible individuals by adopting the framework proposed in [22].

REFERENCES


