Existence of Solutions for a Class of Kirchhoff-type Equation with Nonstandard Growth

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Abstract This paper is concerned with the existence of solutions to a class $p(x)$-Kirchhoff type problem with Dirichlet boundary data. Using a direct variational approach and the theory of the variable exponent Lebesque-Sobolev spaces, we establish some conditions that ensure the existence of nontrivial weak solutions.

Keywords Variational Method, $p(x)$-Kirchhoff Type Equation, Critical Point, Mountain-Pass Theorem, Ekeland Variational Principle

MSC: 35D05;35J60;35J70;58E05

1 Introduction

In this paper, we are concerned with the following Kirchhoff type problem

\[
\begin{cases}
-M (A (x, \nabla u)) \text{div} (a (x, \nabla u)) = m(x) |u|^{q(x)-2} u & \text{in } \Omega ,

u = 0 & \text{on } \partial \Omega ,
\end{cases}
\]

(\text{P})

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain; $\lambda > 0$; $p, q \in C (\overline{\Omega})$ for any $x \in \overline{\Omega}$; $m$ is a nonnegative measurable real function and $\text{div}(a (x, \nabla u))$ is a $p (x) - $Laplace type operator. Moreover, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

Problem (P) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff \cite{16}. To be more precise, Kirchhoff established a model given by the equation

\[
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0 ,
\]

(1.1)

where $\rho, P_0, h, E, L$ are constants, which extends the classical D’Alambert’s wave equation, by considering the effects of the changes in the length of the strings during the vibrations. Moreover, nonlocal boundary value problems like (1.1) can be used for modelling several physical and biological systems where $u$ describes a process which depend on the average of itself, such as the population density \cite{1, 3, 6, 15}.

Recently, the studies of the Kirchhoff equations and the Kirchhoff systems have been considered by variational method in the case involving the $p-$Laplacian operator \cite{4, 7, 8, 18, 19, 21}. Moreover, due to the increasing amount of attention towards partial differential equations with nonstandard growth conditions, it was further extended to the $p(x)-$Laplacian operator $\Delta_{p(x)}$, defined by $\Delta_{p(x)} u := \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right)$ \cite{5, 9, 10, 25, 26}. The $p(x)-$Laplacian possesses more complicated nonlinearities than $p-$Laplacian, for instance, it is not homogeneous. The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications, for instance, the image restoration or the motion of the so called electrorheological fluids, characterized by their ability to drastically change their mechanical properties under the influence of an exterior electromagnetic field \cite{2, 14, 20, 27}.

This paper is organized as follows. In Section 2, we present some necessary preliminary results. In Section 3, using the variational method, we give the existence results of weak solutions of problem (P).
2 Preliminaries

We recall in what follows some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces \( L^{p(x)}(\Omega) \), \( W^{1,p(x)}(\Omega) \) and \( W^{1,0}_{0,p(x)}(\Omega) \). In this context we refer to [13, 17, 22, 24] for the fundamental properties of these spaces.

Set \( C_+((\Omega)) = \{ p; p \in C((\Omega)), \min \, p(x) > 1, \forall x \in (\Omega) \}. \)

For any \( p(x) \in C_+((\Omega)) \), we denote \( 1 < p^- := \min_{x \in (\Omega)} p(x) \leq p(x) \leq p^+ := \max_{x \in (\Omega)} p(x) < \infty \), and define the variable exponent Lebesgue space by

\[
L^{p(x)}((\Omega)) = \left\{ u | u : \Omega \rightarrow \mathbb{R} \text{ is measurable, } \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \right\}.
\]

We define a norm, the so-called Luxemburg norm, on this space by the formula

\[
|u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|u(x)|^{p(x)}}{\lambda} \, dx \leq 1 \right\}
\]

and \( (L^{p(x)}((\Omega)), |.|_{p(x)}) \) becomes a Banach space.

**Proposition 2.1** [13, 17] The conjugate space of \( L^{p(x)}((\Omega)) \) is \( L^{p'(x)}((\Omega)) \), where \( \frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \). For any \( u \in L^{p(x)}((\Omega)) \) and \( v \in L^{p'(x)}((\Omega)) \), we have

\[
\int_{\Omega} uv \, dx \leq \left( \frac{1}{p^+} + \frac{1}{p'^+} \right) |u|_{p(x)} |v|_{p'(x)}. \tag{2.1}
\]

The modular of the \( L^{p(x)}((\Omega)) \) space, which is the mapping \( \rho_{p(x)} : L^{p(x)}((\Omega)) \rightarrow \mathbb{R} \) defined by

\[
\rho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} \, dx, \forall u \in L^{p(x)}((\Omega)).
\]

**Proposition 2.2** [13, 17] If \( u, u_n \in L^{p(x)}((\Omega)), (n = 1, 2, \ldots) \) and \( p^+ < \infty \), we have

\[
(i) \quad |u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}; \tag{2.2}
\]

\[
(ii) \quad |u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}; \tag{2.3}
\]

\[
(iii) \quad \lim_{n \to \infty} |u_n|_{p(x)} = 0 \iff \lim_{n \to \infty} \rho_{p(x)}(u_n) = 0; \tag{2.4}
\]

We consider the weighted variable exponent Lebesgue space \( L^{p(x)}_{c(x)}((\Omega)) \). Let \( c : \mathbb{R}^N \to \mathbb{R} \) be a measurable real function such that \( c(x) > 0 \) a.e. \( x \in (\Omega) \). We define

\[
L^{p(x)}_{c(x)}((\Omega)) = \left\{ uu : \Omega \rightarrow \mathbb{R} \text{ is measurable, } \int_{\Omega} c(x) |u(x)|^{p(x)} \, dx < \infty; \ c(x) > 0 \right\},
\]

with the norm

\[
|u|_{L^{p(x)}_{c(x)}((\Omega))} := |u|_{c(x),p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} c(x) \frac{|u(x)|^{p(x)}}{\lambda} \, dx \leq 1 \right\},
\]

then \( L^{p(x)}_{c(x)}((\Omega)) \) is a Banach space which has similar properties with the usual variable exponent Lebesgue spaces. The modular of this space is \( \rho_{(p(x),c(x))} : L^{p(x)}_{c(x)}((\Omega)) \rightarrow \mathbb{R} \) defined by

\[
\rho_{(p(x),c(x))}(u) = \int_{\Omega} c(x) |u(x)|^{p(x)} \, dx.
\]

**Proposition 2.3** [11] Let \( p(x) \) and \( q(x) \) be measurable functions such that \( p(x) \in L^\infty(\Omega) \), and \( 1 \leq p(x)q(x) \leq \infty \), for a.e. \( x \in (\Omega) \). Let \( u \in L^{q(x)}((\Omega)) \), \( u \neq 0 \). Then

\[
|u|_{p(x)q(x)} \leq 1 \implies |u|_{p(x)q(x)}^{p^+} \leq |u|_{p(x)q(x)}^{p^-} \leq |u|_{p(x)q(x)}^{p^+} \leq |u|_{p(x)q(x)}^{p^-} \leq |u|_{p(x)q(x)}^{p^-} \leq |u|_{p(x)q(x)}^{p^+} \quad \text{(2.5)}
\]

\[
|u|_{p(x)q(x)} \geq 1 \implies |u|_{p(x)q(x)}^{p^-} \leq |u|_{p(x)q(x)}^{p^+} \leq |u|_{p(x)q(x)}^{p^-} \leq |u|_{p(x)q(x)}^{p^-} \leq |u|_{p(x)q(x)}^{p^+} \quad \text{(2.6)}
\]
In particular, if \( p(x) = p \) is constant then \( \|u\|_{l,p(x)}^{p} = |u|_{p,q(x)}^{p} \).

The variable exponent Sobolev space \( W^{1,p(x)}(\Omega) \) is defined by

\[
W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega) \right\},
\]

with the norm

\[
\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega).
\]

The space \( W^{1,p(x)}_{0}(\Omega) \) is denoted by the closure of \( C_{0}^{\infty}(\Omega) \) in \( W^{1,p(x)}(\Omega) \) with respect to the norm \( \|u\|_{1,p(x)} \). We can define \( \|u\| = |\nabla u|_{p(x)} \) for \( u \in W^{1,p(x)}_{0}(\Omega) \).

**Proposition 2.4** [13, 17]

(i) If \( 1 < p^{-} < p^{+} < \infty \), then the spaces \( L^{p(x)}(\Omega) \), \( W^{1,p(x)}(\Omega) \) and \( W^{1,p(x)}_{0}(\Omega) \) are separable and reflexive Banach spaces.

(ii) Let \( q \in C_{+}(\overline{\Omega}) \). If \( q(x) < p^{+}(x) \), for all \( x \in \overline{\Omega} \), then the embedding \( W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \) is compact and continuous, where \( p^{+}(x) = \frac{Np(x)}{N-p(x)} \) if \( N > p(x) \) or \( p^{+}(x) = \infty \) if \( N \leq p(x) \).

### 3 The main results

Let \( X \) denote the variable exponent Sobolev space \( W^{1,p(x)}_{0}(\Omega) \). The energy functional corresponding to problem \( (P) \) is defined as \( I_{\lambda} : X \rightarrow \mathbb{R} \),

\[
I_{\lambda}(u) = \tilde{M} \left( \int_{\Omega} A(x, \nabla u) \, dx \right) - \lambda \int_{\Omega} \frac{m(x)}{q(x)} |u|^{q(x)} \, dx := \tilde{M}(\Lambda(u)) - \lambda J(u),
\]

where \( \tilde{M}(t) = \int_{0}^{t} M(s) \, ds \), \( \Lambda(u) = \int_{\Omega} A(x, \nabla u) \, dx \) and \( J(u) = \int_{\Omega} \frac{m(x)}{q(x)} |u|^{q(x)} \, dx \).

We say that \( u \in X \) is a weak solution of \((1.1)\) if

\[
M \left( \int_{\Omega} A(x, \nabla u) \right) \int_{\Omega} a(x, \nabla u) \nabla \varphi \, dx = \int_{\Omega} m(x) |u|^{q(x)-2} u \varphi \, dx,
\]

for all \( \varphi \in X \), where \( a(x, \xi) : \Omega \times \mathbb{R}^{N} \rightarrow \mathbb{R}^{N} \) is continuous derivative with respect to \( \xi \) of the mapping \( A : \Omega \times \mathbb{R}^{N} \rightarrow \mathbb{R}^{N} \), \( A = A(x, \xi) \), i.e. \( a(x, \xi) = \nabla_{\xi} A(x, \xi) \).

In this article, we assume the following hypotheses:

(A1) The following inequality holds

\[
|a(x, \xi)| \leq c_{0}(h_{0}(x) + |\xi|^{p(x)-1}), \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^{N},
\]

where \( h_{0}(x) \in L^{p(x)}(\Omega) \) is a nonnegative measurable function.

(A2) \( A \) is \( p(x) \)-uniformly convex: There exists a constant \( k > 0 \) such that

\[
A(x, \frac{\xi + \psi}{2}) \leq \frac{1}{2} A(x, \xi) + \frac{1}{2} A(x, \psi) - k |\xi - \psi|^{p(x)}, \text{ for all } x \in \Omega \text{ and } \xi, \psi \in \mathbb{R}^{N}.
\]

(A3) The following inequalities hold true

\[
|\xi|^{p(x)} \leq a(x, \xi) \cdot \xi \leq p(x) A(x, \xi), \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^{N}.
\]

(A4) \( A(x, 0) = 0 \), for all \( x \in \Omega \).

(A5) \( A(x, -\xi) = A(x, \xi) \), for all \( x \in \Omega \) and \( \xi \in \mathbb{R}^{N} \).

(M1) (Polynomial growth condition) \( M : \mathbb{R}^{+} \rightarrow \mathbb{R}^{+} \) is a continuous function and satisfies the condition,

\[
m_{0}s^{\alpha-1} \leq M(s) \leq m_{1}s^{\alpha-1},
\]

for all \( s > 0 \) and \( m_{0}, m_{1} \) real numbers such that \( 0 < m_{0} \leq m_{1} \) and \( \alpha \geq 1 \).

**Lemma 3.1** [23]

(i) \( A \) verifies the growth condition \( \left| A(x, \xi) \right| \leq c_{0}(h_{0}(x) + |\xi|^{p(x)}) \), for all \( x \in \Omega \) and \( \xi \in \mathbb{R}^{N} \);

(ii) \( A \) is \( p(x) \)-homogeneous, \( A(x, z\xi) \leq A(x, \xi) z^{p(x)} \) for all \( z \geq 1, \xi \in \mathbb{R}^{N} \) and \( x \in \Omega \).

**Lemma 3.2**

(i) The functional \( A \) is well-defined on \( X \);
(ii) The functional $\Lambda$ is of class $C^1(X, \mathbb{R})$ and
\[
\langle \Lambda' (u), v \rangle = \int_{\Omega} a (x, \nabla u) \cdot \nabla v dx, \quad \text{for all } u, v \in X;
\]

(iii) The functional $\Lambda$ is weakly lower semi-continuous on $X$;

(iv) For all $u, v \in X$
\[
\Lambda (\frac{u + v}{2}) \leq \frac{1}{2} \Lambda (u) + \frac{1}{2} \Lambda (v) - k \| u - v \|^p - ;
\]

(v) For all $u, v \in X$
\[
\Lambda (u) - \Lambda (v) \geq \langle \Lambda' (v), u - v \rangle ;
\]

(vi) $I$ is weakly lower semi-continuous on $X$;

(vii) $I$ is well-defined on $X$ and of class $C^1(X, \mathbb{R})$, and its derivative given by
\[
\langle I'_u (u), v \rangle = M \left( \int_{\Omega} A (x, \nabla u) dx \right) \int_{\Omega} a (x, \nabla u) \nabla v dx - \int_{\Omega} m (x) |u|^{q(x)} - 2 uv dx ;
\]
for all $u, v \in X$.

Since the proof of Lemma 3.2 is very similar to the proof of Lemma 2.2 and Lemma 2.7 given in [23], we omit it.

**Theorem 3.3** Assume (M1) and the following conditions hold:

(A) $m \in L^{\beta(x)} (\Omega)$, $m (x) > 0$ for $x \in \Omega$, $\beta \in C_+ (\overline{\Omega})$, such that $\frac{1}{\beta (x)} + \frac{1}{\beta_0 (x)} = 1, 1 < p (x) < \beta_0 (x) q (x)$ and $1 < q (x) < \frac{1}{\beta_0 (x)} p^* (x), \forall x \in \Omega$.

(B) $1 < q^{-} < \alpha p^{-} < q^{+}, p^{+} < N$.

Then, there exists $\lambda^* > 0$ such that (P) has a nontrivial weak solution for any $\lambda \in (0, \lambda^*)$.

**Lemma 3.4** Suppose (M1), (A) and (B) hold. Then, there exist two positive real numbers $\gamma, r$ and $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, we have $I_{\lambda} (u) \geq r > 0, u \in X$ with $\| u \| = \gamma$.

**Proof.** By the conditions (A) and the compact embedding from $X$ to the weighted space $L_{m (x)}^{q (x)} (\Omega)$ (see [22], Theorem 2.8), there exists positive constant $c$ such that
\[
\int_{\Omega} m (x) |u|^{q(x)} dx \leq c (\| u \|^{q^{+}} + \| u \|^{q^{-}})^{1 - \delta}, \tag{3.1}
\]
for all $u \in X$. Consider $\| u \| < 1$. Then, by using (M1), (A3), (A4) and (3.1), we can write
\[
I_{\lambda} (u) \geq \frac{m_0}{\alpha} \left( \int_{\Omega} A (x, \nabla u) dx \right)^{\alpha} - \frac{\lambda c}{q^{+}} \int_{\Omega} m (x) |u|^{q(x)} dx \\
\geq \frac{m_0}{\alpha (p^{+})^{q^{+}}} \| u \|^{\alpha p^{+}} - \frac{\lambda c_1}{q^{-}} \| u \|^{q^{-}} \\
= \left( \frac{m_0}{\alpha (p^{+})^{q^{+}}} \right)^{\alpha p^{+} - q^{-}} \gamma^{q^{-}} \tag{3.2}
\]
By the last inequality above, if we choose
\[
\lambda^* = \frac{m_0 q^{-} \rho^{q^{+} - q^{-}}}{2c_1 (p^{+})^{q^{+}}},
\]
then for any $\lambda \in (0, \lambda^*)$ and $\forall u \in X$ with $\| u \| = \gamma$ there exists $r > 0$ such that $I_{\lambda} (u) \geq r > 0$.

The proof of Lemma 3.4 is complete.

**Lemma 3.5** Assume that (M1), (A) and (B) hold. Then, there exist $\omega \in X$ such that $\omega \geq 0, \omega \neq 0$ and $I_{\lambda} (t \omega) < 0$ for $t > 0$ small enough.

**Proof.** We know that $q^{-} < \alpha p^{-}$ from assumption (B). Let $\epsilon_0 > 0$ be such that $q^{-} + \epsilon_0 < \alpha p^{-}$. On the other hand, since $q \in C_+ (\overline{\Omega})$ it follows that there exists an open set $\Omega_0 \subseteq \Omega$ such that $| q (x) - q^{-} | < \epsilon_0$ for all $x \in \Omega_0$. Hence, we conclude that $q (x) \leq q^- + \epsilon_0 < \alpha p^{-}$ for all $x \in \Omega_0$.

Let $\omega \in C_0^\infty (\Omega)$ be such that $\operatorname{supp} (\omega) \supset \Omega_0, \omega (x) = 1$ for all $x \in \Omega_0$ and $0 \leq \omega (x) \leq 1$ in $\Omega$. Then, by the above information for any $t \in (0, 1)$ we have
\[
I_{\lambda} (t \omega) = \int_{\Omega} A (x, \nabla t \omega) dx - \lambda \int_{\Omega} \frac{m (x)}{q (x)} |t \omega|^{q(x)} dx \\
\leq \frac{m_1 \rho^{p^+}}{\alpha} \left( \int_{\Omega} A (x, \nabla \omega) dx \right)^{\alpha} - \lambda \frac{\rho^{q^{-} + \epsilon_0}}{q^{-}} \int_{\Omega} m (x) |\omega|^{q(x)} dx.
\]
Thus
\[ I_\Lambda (t) < 0, \]
for \( t < \delta^{1/(\alpha p-q-\epsilon) \alpha} \) with
\[ 0 < \delta < \min \left\{ 1, \frac{\lambda \alpha}{m_1 q r} \left( \int_{\partial \Omega} m(x) |\omega|^q dx \right)^{1/q} \right\}. \]

**Conclusion (Proof of Theorem 3.3)** By Lemma 3.4, we infer that there exists on the boundary of the ball centered at the origin and of radius \( \rho \) in \( X \) such that \( \inf_{\partial B(0)} I_\Lambda > 0 \). Moreover, from Lemma 3.5, there exists \( \omega \in X \) such that \( I_\Lambda (t \omega) < 0 \), for all \( t > 0 \) small enough. Thus, take into account inequality (3.2), we obtain the following
\[ -\infty < \epsilon := \inf_{B_\rho(0)} I_\Lambda < 0. \]
Let choose \( \epsilon > 0 \). Then, it follows
\[ 0 < \epsilon < \inf_{\partial B_\rho(0)} I_\Lambda - \inf_{B_\rho(0)} I_\Lambda. \]
Applying Ekeland’s variational principle [12] to the functional \( I_\Lambda : B_\rho(0) \to \mathbb{R} \), we can find \( u_\epsilon \in B_\rho(0) \) such that
\[ I_\Lambda (u_\epsilon) < \inf_{B_\rho(0)} I_\Lambda + \epsilon \]
\[ I_\Lambda (u_\epsilon) < I_\Lambda (u) + \epsilon \| u - u_\epsilon \|, \ u \neq u_\epsilon. \]
By the fact that
\[ I_\Lambda (u_\epsilon) < \inf_{B_\rho(0)} I_\Lambda + \epsilon < \inf_{\partial B_\rho(0)} I_\Lambda + \epsilon < \inf_{\partial B_\rho(0)} I_\Lambda, \]
we can infer that \( u_\epsilon \in B_\rho(0) \). Now, we define \( \phi_\Lambda : B_\rho(0) \to \mathbb{R} \) by \( \phi_\Lambda (u) = I_\Lambda (u) + \epsilon \| u - u_\epsilon \| \). It is clear that \( u_\epsilon \) is a minimum point of \( \phi \), and thus
\[ \frac{\phi_\Lambda (u_\epsilon + tv) - \phi_\Lambda (u_\epsilon)}{t} \geq 0, \]
for \( t > 0 \) small enough and any \( v \in B_1(0) \). By the above relation, we have
\[ I_\Lambda (u_\epsilon + tv) - I_\Lambda (u_\epsilon) + \epsilon \| v \| \geq 0. \]
Letting \( t \to 0 \), we have that \( \langle \langle I_\Lambda (u_\epsilon), v \rangle + \epsilon \| v \| > 0 \) and we infer that \( \| I_\Lambda (u_\epsilon) \| \leq \epsilon \). We show that there exists a sequence \( \{ u_n \} \subset B_\rho(0) \) such that
\[ I_\Lambda (u_n) \to c = \inf_{B_\rho(0)} I_\Lambda < 0 \text{ and } I_\Lambda ' (u_n) \to 0. \tag{3.3} \]
Since the sequence \( \{ u_n \} \) is bounded in \( X \), there exists \( u \in X \) such that, up to a subsequence, again denoted by \( \{ u_n \} \), and \( u \in X \) such that \( u_n \to u \) in \( X \) so \( I_\Lambda ' (u_n) , u_n - u \to 0 \) as \( n \to \infty \). On the other hand, we have
\[ \langle I_\Lambda ' (u_n) , u_n - u \rangle = M \left( \int_{\Omega} A (x , \nabla u_n) (\nabla u_n - \nabla u) \, dx - \int_{\Omega} m (x) |u_n|^q - 2 u_n (u_n - u) \, dx \right) \]
Using (A), (2.1) and Proposition 2.4, we get that \( X \) is compact embedding in \( L_{m(x)}^q (\Omega) \) (see [22], Theorem 2.8). Therefore, we obtain that
\[ \lim_{n \to \infty} \int_{\Omega} m (x) |u_n|^q - 2 u_n (u_n - u) \, dx = 0. \]
Using the above information we can write \( \int_{\Omega} a (x , \nabla u_n) (\nabla u_n - \nabla u) \, dx = 0 \), that is, \( \lim_{n \to \infty} \langle A' (u_n) , u_n - u \rangle = 0 \). Moreover, by using Lemma 3.2 (iv), we have
\[ 0 = \lim_{n \to \infty} \langle A' (u_n) , u - u_n \rangle \leq \lim_{n \to \infty} \langle A (u) - A (u_n) \rangle = A (u) - \lim_{n \to \infty} A (u_n) \]
or \( \lim_{n \to \infty} A (u_n) \leq A (u) \). This fact and Lemma 3.2 (iii) imply \( \lim_{n \to \infty} A (u_n) = A (u) \).
We assume by contradiction that \( u_n \) does not converge strongly to \( u \) in \( X \). Then, there exists \( \epsilon > 0 \) and a subsequence \( \{ u_{n_m} \} \) of \( \{ u_n \} \) such that \( \| u_{n_m} - u \| \geq \epsilon \). On the other hand, by in Lemma 3.2 (iv), we have
\[ \frac{1}{2} \Lambda(u) + \frac{1}{2} \Lambda(u_{n_m}) - \Lambda(\frac{u_{n_m} + u}{2}) \geq k \| u_{n_m} - u \|^{p^*} \geq k \varepsilon^{p^*}. \]

Letting \( m \to \infty \) in the above inequality, we obtain

\[ \lim_{n \to \infty} \sup \Lambda(\frac{u_{n_m} + u}{2}) \leq \Lambda(u) - k \varepsilon^{p^*}. \]

Moreover, we have \( \{ \frac{u_{n_m} + u}{2} \} \) converges weakly to \( u \) in \( X \). Using Lemma 3.2 (iii), we have

\[ \Lambda(u) \leq \lim_{n \to \infty} \inf \Lambda(\frac{u_{n_m} + u}{2}), \]

and that is a contradiction. It follows that \( \{ u_n \} \) converges strongly to \( u \) in \( X \). We conclude that \( u \) is a nontrivial weak solution for problem (P). The proof of Theorem 3.3 is complete. □

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