Gaussian Beam Propagation through a Metamaterial Lens

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Abstract

We study a Gaussian beam propagation through a metamaterial lens by direct numerical simulations using COMSOL. We find that a metamaterial lens can deflect the beam significantly by either adjusting the shape of the lens or increasing the dielectric permittivity of the metamaterials.

Keywords

Gaussian Beam, Metamaterial Lens

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1. Introduction

Metamaterials are artificially structured materials with unusual and exciting electromagnetic and optical properties that are not generally found in nature (Cai and Shalaev 2010; Chen, Chan and Sheng 2010; Kildishev and Shalaev 2011; Noginov and Podolskiy 2012). These micro-structured metamaterials have provided a wide array of potential interesting applications. Some examples include invisibility cloaks, directive and omnidirectional antennas, Luneberg and Eaton lenses, waveguide tapers, photonic band gap structures, double negative media (i.e. media having both negative permittivity and negative permeability), and shielding structures from earthquake.

Understanding the propagation of a Gaussian beam through a metamaterial lens is important since it has many potential applications. In particular, by tuning the lens shapes and adjusting material properties one can hope to propagate the beam as desired. In this paper we directly apply numerical techniques to understand the propagation of a Gaussian beam through a metamaterial lens and explore the effects of various parameters in the design of the metamaterial lens. Our main focus here is to study how to control the propagation of the beam by adjusting the metamaterial lens.

Following (COMSOL 2013), we consider a two-dimensional metamaterial lens enclosed in a square air domain which is surrounded by a perfectly matched layer (PML) on each side, as illustrated in Figure 1. A PML is an artificial absorbing layer used to truncate unbounded computational regions in numerical methods for wave equations so that waves incident upon the PML are absorbed without reflection at the interface. Assume a Gaussian beam enters the domain from the left side, via a surface current excitation at an interior boundary given by

\[ J_0 = A e^{-\frac{(x^2+y^2)}{a}} \]  

where \( a \) is the beam waist size and \( A \) measures the beam power. The excitation is at the boundary between PML and the modeling domain, and produces a wave that propagates in both directions – into the PML and into the modeling domain. The wave traveling into the PML is completely absorbed by the PML whereas the wave traveling into the modeling domain is diffracted by the lens. Our main focus here is to study how to control the propagation of the beam by adjusting the metamaterial lens.

![Figure 1. A metamaterial lens enclosed in an air domain.](image)

The real shape of the metamaterial lens is defined as functions of the Cartesian coordinates \((x_u, y_u)\) of the undeformed rectangular frame:

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**Figure 1.** A metamaterial lens enclosed in an air domain.
\[ x = x_u \left( a_1 + a_2 y_u^2 \right), \quad y = y_u \left( a_3 + a_4 x_u^2 \right) \quad (2) \]

where the typical values for the parameters are \( a_1 = 1, a_2 = -0.5, a_3 = 1, a_4 = 0.5 \). The dielectric distribution of the metamaterial lens is defined on the original Cartesian domain by the relationship

\[ \varepsilon_r = \left( b_1 + b_2 y_u^2 \right)^2 \quad (3) \]

which leads to a spatial variation in the dielectric distribution on the deformed lens. The typical values for the parameters are \( b_1 = 1, b_2 = 0.5 \). The special case where \( b_2 = 0 \) corresponds to the situation without a metamaterial lens.

The left boundary of the lens (shown in Figure 2) where the beam enters the lens is specified by the curve

\[ x = -\left( a_1 + a_2 y_u^2 \right), \quad y = y_u \left( a_3 + a_4 \right) \]

whose curvature is

\[ \kappa = \frac{\left[ x y_u^2 - y x_u^2 \right]}{\left[ x^2 + y_u^2 \right]^{3/2}} = \frac{\left[ \left( -a_1 + a_4 \right)^2 + \left( a_3 + a_4 \right)^2 \right]^{3/2}}{4 \left[ a_2 \left( a_3 + a_4 \right) \right]^{3/2}} \quad (4) \]

where the primes indicate the derivatives with respect to \( y_u \). So the curvature depends on the spatial variable \( y_u \) and the parameters \( a_2 \) and \( a_3 + a_4 \).

![Figure 2. The left boundary of a metamaterial lens where the beam enters the lens.](image)

From (3) and (4) we see that the dielectric distribution of the metamaterial lens is inversely proportional to the curvature of the interface of the lens.

In a flat three-dimensional Euclidean space, the beam propagation through the lens is described by the macroscopic Maxwell’s equations (Yeh, 2005):

\[ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \]

\[ \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \]

\[ \nabla \cdot \vec{D} = \rho \]

\[ \nabla \cdot \vec{B} = 0 \]

The first two equations are also called Faraday’s law and Maxwell-Ampere’s law, respectively. Equation three and four are the electric and magnetic form from of Gauss’s law, respectively. In these equations, \( \vec{E} \) denotes the electric field vector while \( \vec{H} \) represents the magnetic field vector. The quantities \( \vec{D} \) and \( \vec{B} \) are the electric displacement (or electric flux density) and the magnetic induction (or magnetic flux density), respectively. The remaining quantities are the electric current density \( \vec{J} \) and the electric charge density \( \rho \). Maxwell’s equations contain eight scalar equations that involve twelve variables. To obtain a closed system, Maxwell’s equations are supplemented by the constitutive relations that describe the macroscopic properties of the medium:

\[ \vec{D} = \varepsilon_r \varepsilon_0 \vec{E} + \vec{P} \]

\[ \vec{B} = \mu_r \mu_0 \vec{H} + \vec{M} \quad (6) \]

Here \( \varepsilon_r \) is the dielectric or permittivity tensor, and \( \mu_r \) is the permeability tensor of the material. The constant \( \varepsilon_0 \) is the permittivity of a vacuum and has a value of \( 8.854 \times 10^{-12} \text{F/m} \), \( \mu_0 \) is the permeability of a vacuum and has a value of \( 4\pi \times 10^{-7} \text{H/m} \). The electric polarization vector \( \vec{P} \) describes how the material is polarized when an electric field is present in matter. Similarly, the magnetization vector \( \vec{M} \) describes how the material is magnetized when a magnetic field is present. For linear materials, the polarization is directly proportional to the electric field and the magnetization is directly proportional to the magnetic field. For such materials, the constitutive equations become

\[ \vec{D} = \varepsilon_r \varepsilon_0 \vec{E} \]

\[ \vec{B} = \mu_r \mu_0 \vec{H} \quad (7) \]

where the parameter \( \varepsilon_r \) is the relative permittivity and \( \mu_r \) is the relative permeability of the material. These are usually scalars if the material medium is isotropic, and tensors for anisotropic material.

If we substitute the constitutive relation (6) for \( \vec{B} \) into Faraday’s law in (4), divide both sides by \( \mu_0 \mu_r \), and apply the curl operator, we get

\[ \nabla \times \left( \mu_0^{-1} \mu_r^{-1} \nabla \times \vec{E} \right) + \frac{\partial}{\partial t} \nabla \times \vec{H} = \vec{0} \quad (8) \]

Now differentiating Maxwell-Ampere’s law in (4) with respect to time and combining it with (7), we find...
\[ \nabla \times (\mu_r^{-1} \mu_r \nabla \times \vec{E}) + \frac{\partial}{\partial t} \left( \frac{\partial \vec{D}}{\partial t} + \vec{J} \right) = \vec{0} \quad (9) \]

Using the material equation (6) and assuming \( \vec{J} = \sigma \vec{E} \) with \( \sigma \) being the electric conductivity, we obtain
\[ \nabla \times (\mu_r^{-1} \mu_r \nabla \times \vec{E}) + \epsilon_0 \epsilon_r \frac{\partial^2 \vec{E}}{\partial x^2} + \sigma \frac{\partial \vec{E}}{\partial t} = \vec{0} \quad (10) \]

We consider the time-harmonic electromagnetic fields where the electric field can be written as
\[ \vec{E}(r, t) = \text{Re} \left( \vec{E}(r) e^{i\omega t} \right) \quad (11) \]

Here \( \vec{E}(r) \) is a phase vector, or phasor whose amplitude and phase are time-invariant, and \( \omega \) is the angular frequency. Note that the time derivative of (11) corresponds to a multiplication by a simple factor \( i \omega \), namely,
\[ \frac{\partial}{\partial t} \vec{E}(r, t) = \text{Re} \left( i \omega \vec{E}(r) e^{i\omega t} \right) \quad (12) \]

Employing the property (12), we can replace the time-dependent equation (9) by a time-independent equation for the phasor:
\[ \nabla \times (\mu_r^{-1} \mu_r \nabla \times \vec{E}) + \epsilon_0 \epsilon_r (i \omega)^2 \vec{E} + \mu_0 \sigma (i \omega) \vec{E} = \vec{0} \quad (13) \]

or equivalently,
\[ \nabla \times (\mu_r^{-1} \mu_r \nabla \times \vec{E}) - k_0^2 \left( \epsilon_r - \frac{i \omega}{\epsilon_0 \omega} \right) \vec{E} = \vec{0} \quad (14) \]

where the free space wave number \( k_0 \) is given by \( k_0 = \omega / c = \omega \epsilon_0 \mu_0 \) and the speed of light is \( c = 1 / \sqrt{\epsilon_0 \mu_0} \).

### 3. Numerical Results

In this section we investigate the effects of various parameters on the propagation of a Gaussian beam through a metamaterial lens. We apply the commercial software COMSOL to solve the partial differential equation (14) with the prescribed initial and boundary conditions. COMSOL is a multi-physics commercial software based on finite element methods which can be used to solve our model equations efficiently.

#### 3.1. Beam Effects

First we fix the properties of a metamaterial lens and study the propagation of a Gaussian beam and how it is affected by various beam characteristics.

In Figure 3 we compare the numerical solutions with and without a metamaterial lens. The parameters used here are \( A = 1, f_0 = 3 \text{GHz}, a = 2(c / f_0) \) where \( f_0 \) represents the operating frequency and \( a \) is the Gaussian beam waist size. Here the beam waist size is inversely proportionally to the operating frequency. All the other parameters take the typical values specified earlier. In Figure 3(a) we plot the norm of the electric field through a medium of air. In contrast, in Figure 3(b) we plot the norm of the electric field passing through a metamaterial lens. Figure 3(c) shows the difference between the two cases in Figure 3(b) and Figure 3(a). In Figure 3(d) we illustrate the distribution of the electric field norm along a vertical line where \( x = 0.5 \).

Without a lens the maximum value of the electric field norm is about 140 V/m and it is reduced to 119 V/m by the metamaterial lens, which is about 15% reduction.
Now we investigate the effect of the beam waist size. We double the beam waist size to $a = 4(\epsilon / f_o)^2$ and give our results in Figure 4. Figure 4(a) corresponds to the beam propagation in air while Figure 4(b) is related to the beam propagation through a metamaterial lens. Figure 4(c) shows their difference. The metamaterial lens has deflected the beam significantly. Figure 4(d) gives the distribution of the electric field norm along the vertical line $x = 0.5$. The maximum norm of the electric field is decreased from 182 V/m without a lens to 150 V/m with a metamaterial lens, which is about 17% reduction.

In Figure 5 we decrease the beam waist size to $a = (\epsilon / f_o)^2$ and show the results similar to Figures 3-4. As shown in Figure 5(d), the maximum value of the electric field norm decreases from 76 V/m to 66 V/m with a metamaterial lens. This is about 13% reduction in magnitude. Figures 3-5 also suggest that the maximum value of the difference between the case with a lens and the case without a lens is proportional to the beam waist size. Or said differently, the maximum value of the difference between the case with a lens and the case without a lens is inversely proportional to the operating frequency.

In Figure 6 we increase the beam power from $A = 1$ to...
$A = 2$ and keep all the other parameters the same as in Figure 3. A direct comparison between Figure 3 and Figure 6 shows that as the beam power increases, the electric field increases and their relationship is linear. From Figure 6(d) we see that the maximum norm of the electric field is reduced about 15% from 280 V/m to 238 V/m by the metamaterial lens.

From all the above simulations, we can see that the metamaterial lens can diffract the beam and effectively reduce the maximum norm of the electric field by more than 10% in all the cases we have investigated here.
Figure 6. The norm of the electric field with $A = 2$ where the beam waist size $a = 2(c / f_0)^2 = 0.19986 m$. (a) without a lens; (b) with a metamaterial lens; (c) difference between the case with lens and the case without lens; (d) along the vertical line where $x = 0.5$.

3.2. Effects of a Metamaterial Lens Shape

Having investigated the effect of beam parameters, we now move to study the effects of various parameters in the design of a metamaterial lens. All the parameters are the same as in Figure 2 unless specified explicitly.

First, we vary the parameter $a_1$ and obtain various shapes of lens in Figure 7(a) and plot the norm of the electric field along the vertical line $x = 0.5$ for each case in Figure 7(b). The larger value of $a_1$ defines a larger lens and consequently reduces the maximum norm of the electric field more. Due to the change of the shape of the lens, the center of the bell-shaped curve in each case is gradually shifted to the right as the value of $a_1$ increases. In Figure 7(c) we show the dependence of the maximum norm of the electric field along the line $x = 0.5$ on the parameter $a_1$. We find that the numerical data can be fitted by a linear function $\|E\|_{max} = -38.9794a_1 + 158.0745$ very well for the parameter range $0.8 \leq a_1 \leq 1.2$.

Second, we vary the parameter $a_2$. As we change the
values of $a_2$ from negative to positive values, the concavity of the lens changes, as illustrated in Figure 8(a). However, from Figure 8(b) we see that the maximum norm of the electric field is not sensitive to the change of $a_2$.

Next, we vary the parameter $a_3$ and plot the results in Figure 9. As $a_3$ decreases, the lens shape gets shorter and the maximum norm of the electric field decreases as well. Figure 9(c) indicates that the dependence of the maximum norm of the electric field along the line $x = 0.5$ on the parameter $a_3$ can be well approximated by a quadratic function $\|\mathbf{E}\|_{\text{max}} = -39.1799 a_3^2 + 113.2170 a_3 + 44.9701$ for the parameter range $0.8 \leq a_3 \leq 1.2$.

Lastly, we investigate the effect of the parameter $a_4$. Since the variation of $a_4$ does not change the shape of lens that much, the maximum norm of the electric field does not change dramatically, as shown in Figure 10.

**Figure 8.** (a) Various shapes of lens described by different values of $a_2$. (b) The norm of the electric field along the vertical line $x = 0.5$ for various values of $a_2$.

**Figure 9.** (a) Various shapes of lens described by different values of $a_3$. (b) The norm of the electric field along the vertical line $x = 0.5$ for various values of $a_3$. (c) The dependence of the maximum norm of the electric field along $x = 0.5$ on $a_3$. 
To conclude, the design parameters $a_2$ and $a_4$ in Equation (2) do not affect the maximum norm of the electric field that much, whereas varying the design parameters $a_1$ and $a_3$ can lead to significant changes to the maximum norm of the electric field. To reduce the maximum norm of the electric field, one would increase $a_1$ or reduce $a_3$. $a_1$.

Figure 10. (a) Various shapes of lens described by different values of $a_4$. (b) The norm of the electric field along the vertical line $x = 0.5$ for various values of $a_4$.

3.3. Effects of the Dielectric Property of a Metamaterial Lens

We first explore the effect of the parameter $b_1$ in the dielectric distribution of the lens. As shown in Figure 11(a), the maximum norm of the electric field decreases as the value of $b_1$ decreases. Figure 11(b) reveals that the maximum norm of the electric field along the line $x = 0.5$ can be approximated as a function of $b_1$ in the quadratic form

$$|E|_{max} = -75.6331b_1^2 + 191.1888b_1 + 3.1746$$

for the parameter range $0.5 \leq b_1 \leq 1.0$.

Similarly, the effect of the parameter $b_2$ is shown in Figure 12, together with snapshots of the relative permittivity on the lens with $b_2 = 0.2$ and $b_2 = 0.7$. As $b_2$ increases, the maximum norm of the electric field decreases. The maximum norm of the electric field along the line $x = 0.5$ can be fitted as a function of $b_2$ in the quadratic form

$$|E|_{max} = -75.6331b_2^2 + 191.1888b_2 + 3.1746$$

for $0.2 \leq b_2 \leq 0.7$.

From above simulations, we see that in order to reduce the maximum norm of the electric field, one would design a
metamaterial with smaller values of $b_1$ or larger values of $b_2$ in Equation (3). Another possibility is to change the dielectric distribution from Equation (3) to some other forms.

As a preliminary study, we change the form of the dielectric distribution into

$$
\varepsilon_r = \left( b_3 + b_4 x_w^2 \right)^2
$$

(15)

where the default values of the two coefficients are $b_3 = 1, b_4 = 0.5$. Figure 13(a) shows how the norm of the electric field along the vertical line $x = 0.5$ is affected by the variation of $b_3$.

As $b_3$ decreases, the maximum value of the norm of the electric field decreases. Moreover, the maximum norm of the electric field along the line $x = 0.5$ can be represented as a quadratic function of $b_3$:

$$
|E|_{max} = -111.2664 b_3^2 + 243.7007 b_3 + 7.7616
$$

for $0.5 \leq b_3 \leq 1.0$.

Figure 12. (a) The norm of the electric field along the vertical line $x = 0.5$ for various values of $b_2$ . (b) The dependence of the maximum norm of the electric field along $x = 0.5$ on $b_2$ . (c) The contour plot of the dielectric distribution, shown here is the relative permittivity where $b_2 = 0.2$ on the left and $b_2 = 0.7$ on the right.
Figure 13. (a) The norm of the electric field along the vertical line $x = 0.5$ for various values of $b_3$. The dependence of the maximum norm of the electric field along $x = 0.5$ on $b_3$. (c) The contour plot of the dielectric distribution with $b_3 = 1.0$.

In contrast, the effect of varying $b_4$ from 0.2 to 0.7 is almost negligible, as seen in Figure 14.

This observation, together the results shown in Figure 12, also indicates that the inhomogeneity of the dielectric property in the x-direction contributes very little to the reduction of the maximum electric field; whereas the inhomogeneity of the dielectric property in the y-direction can lead to significant decrease in the electric field.

4. Conclusions and Future Work

We have investigated a Gaussian beam propagation in a metamaterial lens, focusing on the effects of the beam, lens shape and dielectric property of the metamaterials. Our findings reveal that it is possible to deflect the beam by either adjusting the shape of the lens or varying the dielectric permittivity. We postpone the search of optimal design of the metamaterial lens shape and dielectric properties to future work.

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