A Note on Domatic Subdivision Stable Graphs

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Abstract  A domatic partition of a graph $G = (V, E)$ is a partition of $V$ into disjoint sets $V_1, V_2, ..., V_k$ such that each $V_i$ is a dominating set for $G$. A subdivision of a graph $G$ is a graph resulting from the subdivision of edges in $G$. In this paper we discuss about the minimal properties of domatic subdivision stable graph and we show that every graph is an induced subgraph of a domatic subdivision stable graph. We discuss methods of generating new domatic subdivision stable graphs from existing domatic subdivision stable graphs using graph operations.

Keywords  Domatic Partition, Subdivision, Dominating Set, Minimal Set

1. Introduction

We consider only simple connected undirected graphs $G = (V, E)$. $P_n$ denotes the path of length $n$. The open neighborhood of vertex $v \in V(G)$ is defined by $N(v) = \{u \in V(G) | uv \in E(G)\}$ while its closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. The private neighborhood of $v \in D$ is defined by $p_n[v, D] = N(v) – N(D – \{v\})$. We say that $H$ is a subgraph of $G$, if $V(H) \subseteq V(G)$ and $uv \in E(H)$ implies $uv \in E(G)$. If a subgraph $H$ satisfies the added property that for every pair $u, v$ of vertices, $uv \in E(H)$ if and only if $uv \in E(G)$, then $H$ is called an induced subgraph of $G$ and is denoted by $\langle H \rangle$. We indicate that $u$ is adjacent to $v$ by writing $u \perp v$. For properties related to graph theory we refer to West, D. B. [1].

A set of vertices $D$ in a graph $G = (V, E)$ is a dominating set if every vertex of $V – D$ is adjacent to some vertex of $D$. If $D$ has the smallest possible cardinality of any dominating set of $G$, then $D$ is called a minimum dominating set — abbreviated MDS. The cardinality of any MDS for $G$ is called the domination number of $G$ and it is denoted by $\gamma (G)$. A $\gamma$-set denotes a dominating set for $G$ with minimum cardinality. A dominating set $D$ is minimal dominating if no proper subset of $D$ is a dominating set. A vertex in $V – D$ is $k$-dominated if it is dominated by at least $k$ vertices in $D$. For properties related to domination we refer to Haynes, T. W. et al., [3, 4].

In [2], Vestergaard, P. D. et al., have studied the domatic numbers and total domatic numbers of graphs having cut vertices. In Theorem 1, 3 and 4 of [2], they have provided upper bounds related to domination numbers and provided conditions for attaining the sharp bounds.

In [6], Yamuna, M. et al., have defined domatic subdivision stable graphs. In Theorem 1, 2 and 3 of [6], they have provided bounds for domatic subdivision stable graphs and obtained necessary and sufficient conditions for the domatic partition of the domatic subdivision stable graphs.

2. Materials and Methods

A domatic partition of a graph $G = (V, E)$ is a partition of $V$ into disjoint sets $V_1, V_2, ..., V_k$ such that each $V_i$ is a dominating set for $G$. The domatic number is the maximum number of such disjoint sets and it is denoted by $d (G)$. A subdivision of a graph $G$ is a graph resulting from the subdivision of edges in $G$. The subdivision of some edge $e$ with endpoints $\{u, v\}$ yields a graph containing one new vertex $w$, and with an edge set replacing $e$ by two new edges, $uw$ and $wv$. We shall denote the graph obtained by subdividing any edge $uv$ of a graph $G$, by $G_{sd} uv$. Let $w$ be a vertex introduced by subdividing $uv$. We shall denote this by $G_{sd} uv = w$.

Yamuna, M and Karthika, K. [6], defined a new graph called domatic subdivision stable graph. A graph $G$ is said to be domatic subdivision stable (dss), if $d (G) = d (G_{sd} uv)$, for all $uv \in E(G)$.

Example of Dss Graph

Figure 1.

G is dss. Here $d (G) = |\{5, 6\}, \{8, 2, 4\}, \{1, 7, 3\}| = 3$ and $d (G_{sd} 58) = |\{4, w, 6\}, \{5, 2, 3\}, \{1, 7, 8\}| = 3$. This is true for all $uv \in E(G)$. Let $S = \{5, 6, 7, 8\} \subseteq V(G)$. $\langle S \rangle = C_4$ in $G$.

Yamuna, M and Karthika, K [6], have proved the following
When \( d(G) = 2 \), \( d(G_{sd uv}) = 2 \) if for all possible partition \( d(G) = |\{U_1\}, \{U_2\}, \{U_3\}| \) such that

- \( V_1 \) and \( V_2 \) are dominating sets for \( G \) such that \( u \in V_1 \), \( v \in V_2 \).
- \( v \) is 2-dominated with respect to \( V_1 \).
- \( u \) is 2-dominated with respect to \( V_2 \).
- \( V_3 \) dominates at least \( G - \{u\} - \{v\} \).

When \( d(G) = 2 \), \( d(G_{sd uv}) = 3 \) if and only if there is a domatic partition \( d(G) = |\{V_1\}, \{V_2\}, \{V_3\}| \) such that \( V_1 \) and \( V_2 \) are both minimal dominating sets for \( G \), \( V_3 \) dominates \( G - \{u\} - \{v\} \), a contradiction to the hypothesis of the theorem.

Case ii \( V_1 \) is not minimal

- Atleast one of \( V_i \) is not minimal.
- \( V_{11}, V_{12} \) are minimal dominating sets for \( G \), where \( V_1 = \{V_{11}, V_{12}\} \) and \( V_2 = \{V_{21}, V_{22}\} \).
- \( V_{12}, V_{22} \) dominates \( G - \{u\} - \{v\} \).

Then there is atleast one partition for \( G \), where \( u, v \) does not belong to the same partition.

Proof

Assume that \( u, v \) belong to the same partition for every possible domatic partition for \( G \), say \( u, v \in V_1 \), where \( d(G) = |\{V_1\}, \{V_2\}| \).

Case i \( V_2 \) is not minimal

- \( V_1 \) is minimal.
- \( V_2 = \{V_{21}, V_{22}\} \), where \( V_{21} \) is a dominating set and \( V_{22} \) dominates \( G - \{u\} - \{v\} \).
- \( u, v \in V_2 \).
- \( V_3 = \{V_{31}, V_{32}\} \), where \( V_{31} \) is minimal dominating set for \( G \) such that \( u, v \in V_3 \), a contradiction.

In all three cases \( d(G) = |\{V_3\}, \{V_4\}| \) is a domatic partition for \( G \) such that \( u, v \in V_4 \), a contradiction.

Hence there is atleast one partition for \( G \), where \( u, v \) does not belong to the same partition.

Theorem 5

Let \( d(G) = 3 \). If for any \( u, v \in E(G) \), there is a domatic partition \( d(G) = |\{V_1\}, \{V_2\}, \{V_3\}| \) such that

- \( u \in V_1, v \in V_2 \).
- \( u \) is selfish with respect to \( V_1 \).
- \( v \) is selfish with respect to \( V_2 \).

Then \( d(G_{sd uv}) = 3 \).

Proof

Let \( d(G) = 3 \) and \( u, v \in V_2 \). Let \( d(G) = |\{V_1\}, \{V_2\}, \{V_3\}| \) such that the conditions of the theorem is satisfied.

Then,

When \( u \in V_1 \), \( v \) is 2-dominated.

When \( v \in V_2 \), \( u \) is 2-dominated.

\( V_2 \) does not dominate \( G - \{u\} - \{v\} \).

By R3, \( d(G_{sd uv}) = 3 \). Also \( d(G_{sd uv}) = |\{V_1\}, \{V_2\}, \{V_3\} | \).
Remark
If G is a graph such that \( d(G) = 3 \) and the conditions of the Theorem 5 is satisfied for all \( uv \in E(G) \), then G is dss.  

Theorem 6
Let \( d(G) = |\{V_1\}, \{V_2\}, \ldots, \{V_k\}|. \) If G is a graph such that \( d(G) = k \) only if exactly one of \( V_i \) is a \( \gamma \)-set, then every \( V_i \), \( i = 1, 2, \ldots, k \) are minimal dominating set.  

Proof
Let \( d(G) = |\{V_1\}, \{V_2\}, \ldots, \{V_k\}| \) be a dominating partition for G satisfying the conditions of the theorem.  
Let \( V_j \) be the \( \gamma \)-set for G. Assume that there is one \( V_{j_1} \neq j, j = 1, 2, \ldots, k \) that is not minimal. Let \( V_j = V_{j_1} \cup V_{j_2} \) such that \( V_{j_2} \) is a minimal dominating set. Then \( V_1, V_2, \ldots, V_j \cup V_{j_2}, \ldots, V_j - V_{j_2}, \ldots, V_k \) is a dominating partition for G such that \( d(G) = k \), which contains no \( \gamma \)-set, a contradiction to our assumption.  

Generating New dss Graphs Using Graph Operations
In this section we show that every graph is an induced subgraph of a dss graph, and discuss two methods of generating new dss graphs from existing dss graphs.  

Theorem 7
Any tree is dss.  

Theorem 8
Let \( G \) be a graph with \( n - \) vertices. Let \( V(G) = \{ v_1, v_2, \ldots, v_n \} \). Consider \( n \) – copies of the complete graph \( K_3: x_i y_i w_i, i = 1, 2, \ldots, n \). Merge \( v_i \) with \( u_i \) and label the new vertex as \( x_i, i = 1, 2, \ldots, n \). We include the following edges.  

\[
\begin{align*}
\forall i, j \neq i, \quad & y_1 y_2, y_2 y_3, \ldots, y_{n-1} y_n, y_n y_1 \quad \text{and} \quad x_1 y_2, x_2 y_3, \ldots, x_{n-1} y_n, x_n y_1,
\end{align*}
\]

That is \( V(\hat{G}) = \{ x_1, x_2, \ldots, x_n \}, Y = \{ y_1, y_2, \ldots, y_n \} \) and \( W = \{ w_1, w_2, \ldots, w_n \} \). As \( P_3: x_i w_i y_i, i = 1, 2, \ldots, n \) are subgraphs of \( H \), By R1, \( d(H) \leq 3 \). Also \( d(H) = |\{X\}, \{Y\}, \{W\}| = 3 \). Let \( d(\hat{G}) = w_i, \) for all \( uv \in E(G) \).

\[
\begin{align*}
d(H) = \left\{ \begin{array}{ll}
\{y_i, x_i, \} & \text{if } i = 1, 2, \ldots, n,
\{w_i, x_i, y_i, \} & \text{if } i = 1, 2, \ldots, n,
\end{array} \right.
\end{align*}
\]

Also \( d(\hat{G}) \) is also a tree. We know that for any graph \( d(G) \leq 2 \). Hence every graph is an induced subgraph of a dss graph \( H \) such that \( d(H) = 3 \).  

Example

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\[
\begin{align*}
\forall i, j \neq i, \quad & y_1 y_2, y_2 y_3, \ldots, y_{n-1} y_n, y_n y_1 \quad \text{and} \quad x_1 y_2, x_2 y_3, \ldots, x_{n-1} y_n, x_n y_1,
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\[
\begin{align*}
d(H) = \left\{ \begin{array}{ll}
\{y_i, x_i, \} & \text{if } i = 1, 2, \ldots, n,
\{w_i, x_i, y_i, \} & \text{if } i = 1, 2, \ldots, n,
\end{array} \right.
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Example

Figure 2. \( G_1 \) is not dss. Also \( < x_1, x_2, \ldots, x_8 > = G_1 \) in \( G_2 \). The induced subgraph is shown in red lines.

Vestergaard, P. D and Zelinka, B. [2], have provided the following result.

Let \( G \) be the union of two graphs \( G_1 \) and \( G_2 \) having exactly one common vertex \( a \); this vertex \( a \) is the cut vertex of \( G \). The graph obtained from \( G_1 \) and \( G_2 \) by deleting a will be denoted respectively by \( G_1' \), \( G_2' \).

R. With the above notation, for every graph \( G \), the domatic numbers satisfy

\[
\begin{align*}
\min \{ d(G), d(G_2) \} \leq d(G) \leq 1 + \min \{ d(G_1'), d(G_2') \}
\end{align*}
\]

They define a saturated vertex as follows

Saturated Vertex
A vertex \( x \) of a graph \( G \) is called saturated, if it is adjacent to all other vertices of \( G \). For example in the complete graph \( K_n \) every vertex is saturated.

Theorem 10
Let \( G_1 \) and \( G_2 \) be two graphs such that \( d(G_1) = d(G_2) = k \). Let \( u \in V(G_1), v \in V(G_2) \) such that \( d(G_1 - \{u\}) = d(G_1'), d(G_2 - \{v\}) = d(G_2') \). Let \( G \) be the graph obtained by merging vertex \( u \) and \( v \). Then \( d(G) = k \).  

Proof
Let \( G_1 \) and \( G_2 \) be two graphs such that \( d(G_1) = d(G_2) = k \).
Let $u \in V(G_1), v \in V(G_2)$ such that $d(G_1) = d(G_2) = k$. Let $u \in V(G_1), v \in V(G_2)$ such that $d(G_1 - \{u\}) = d(G_2) = d(G_1 - \{v\}) = d(G_2)$. Let $G$ be the graph obtained by merging vertex $u$ and $v$.

Claim 1
If $d(G) = d(G - \{u\})$, for some $u \in V(G)$, then $u$ cannot be a saturated vertex.

Proof
If possible assume that $u$ is a saturated vertex. Since $d(G) = d(G - \{u\}) = k$ (say), there are $k - 1$ dominating sets $V_1, V_2, \ldots, V_k$ for $G - \{u\}$. Since $u$ is a saturated vertex, $V_1, V_2, \ldots, V_k$ are dominating sets for $G$ also, which implies $V_1, V_2, \ldots, V_k, u$ are all dominating sets for $G$. This implies $d(G) > k$, a contradiction. So, if $d(G) = d(G - \{u\})$, for any graph $G$, $u$ is not a saturated vertex.

From Claim 1, we conclude that $u$ and $v$ are not saturated vertices with respect to $G_1$ and $G_2$.

Since $uv$ is a cut vertex, we observe that, if $U_1$ is a dominating set for $G$, then $U_1$ can be partition as $U_1 = U_1 \cup \{uv\}$, if $uv \in U_1$.

Also we observe that, $U_1 \cup \{v\}$ are dominating sets for $G_1$ and $G_2$ respectively, if $uv \notin U_1$. Since $d(G_1) = d(G_2) = k$, by R4, $d(G) \leq k + 1$.

Claim 2
$d(G) \neq k + 1$.

Proof
If possible, let $d(G) = k + 1 = |\{U_1\}, \{U_2\}, \ldots, \{U_k\}|$ such that $u \in U_1$. Since $d(G_1) = 3$, let $d(G_1) = |\{C_1\}, \{C_2\}, \{C_3\}|$ such that $v \in C_1$. Then $U_1 \cup C_1 - \{u\} - \{v\} \cup \{uv\}, U_2 \cup C_2, U_3 \cup C_3$ is a domatic partition for $G$, which implies $d(G) > k + 1$, a contradiction.

By claim 2, we conclude that $d(G) = k$.

Converse of Theorem 10 need not be true.

Example

Figure 4. $d(G_1) = d(G_1 - \{xy\}) = 3, xy \in E(G_1)$ and $d(G_2) = d(G_2 - \{xy\}) = 3, xy \in E(G_2)$. Here $d(G) = d(G_{ab}) = 3$, for all $ab \in E(G)$.

Theorem 11
If $G_1$ and $G_2$ are dss graphs such that $d(G_1) = d(G_2) = 3, d(G_1) = d(G_1 - \{u\}), d(G_2) = d(G_2 - \{v\})$, for some $u \in V_1, v \in V_2$, then the graph $G$ obtained by merging $u$ and $v$ is also dss.

Proof
By Theorem 10, we know that $d(G) = 3$. Let $G_{xy} = w$, for some $xy \in E(G)$. Let $e = xy \in E(G)$, either $e \in E(G_1)$ or $e \in E(G_2)$. Assume that $e \in E(G_1)$.

Since $G_1$ is dss, let $d(G_{1-xy}) = |\{U_1\}, \{U_2\}, \{U_3\}|$ such that $u \in U_1$. Since $d(G_2) = 3$, let $d(G_2) = |\{C_1\}, \{C_2\}, \{C_3\}|$ such that $v \in C_1$. Then $U_1 \cup C_1 - \{u\} - \{v\} \cup \{uv\}, U_2 \cup C_2, U_3 \cup C_3$ is a domatic partition for $G_{xy}$, which implies $d(G_{xy}) > 3$. Since $P_3: xy$ is a subgraph of $G$, by R1, we know that $d(G_{xy}) \leq 3$. This implies $d(G_{xy}) = 3$. This is true for every $e \in E(G)$, that is $G$ is dss.

Example

Figure 5. $G$ is a dss graph, $d(G) = d(G_1) = d(G_2) = 3, d(G_1) = d(G_1 - \{3\}) = d(G_2) = d(G_2 - \{6\}) = 2$, that is $d(G_1) \neq d(G_1 - \{3\})$ and $d(G_2) \neq d(G_2 - \{6\})$.

Theorem 12
If $G_1$ and $G_2$ are dss graphs such that $d(G_1) = d(G_2) = 2, d(G_1) = d(G_1 - \{u\}), d(G_2) = d(G_2 - \{v\})$, for some $u \in V_1, v \in V_2$, then the graph $G$ obtained by merging $u$ and $v$ is also dss.

Proof
By Theorem 10, we know that $d(G) = 2$. For any graph $G$,
Assume that \( e = xy \in E(H) \) holds if and only if there is a domatic partition \( d(G) = \{ \{ V_1 \}, \{ V_2 \}, \ldots, \{ V_{i-1} \}, \{ V_i \} \} \), \( x \in V_1, y \in V_2 \) such that

i. \( x \) is 2-dominated with respect to \( V_1 \).

ii. \( V_2 = V_{12} \cup V_{22} \), where

a. \( y \in V_{12}, \) \( y \) is 2-dominated with respect to \( V_{12} \).

b. \( V_{22} \) does not dominate at least one of \( x \) or \( y \), that is \( V_{22} \) is a dominating set for \( G - \{ x \} - \{ y \} \).

\( \forall y \in V_{12}, \) \( \forall y \in V_{22} \).

\( \{ w \} \) is a dominating set for \( G_{sd ab} \), where \( u = u_1 \). \( w \) is 2-dominated. \( U_3 \cap V_1 \). \( U_2 \cap V_2 \).

\( \{ w \} \) is a dominating set for \( G_{sd ab} \), where \( u = u_1 \), \( \forall v \in V_{12}, U_1 \cup V_2 \). \( U_2 \cup V_1 \).

\( \{ w \} \) is a dominating set for \( G_{sd ab} \), where \( u = u_1 \), \( \forall v \in V_{12}, U_1 \cup V_2 \). \( U_2 \cup V_1 \).

From i, ii and iii we conclude that \( d(G) = 3 \). Since \( P_3 \): \( xwy \) is a subgraph of \( G \), by \( R_5 \), we know that \( d(G) = 3 \), which implies \( d(G_{sd ab}) = 3 \).

We conclude that \( d(G) = d(G_{sd ab}) = 3 \), for all \( ab \in E(G) \).

Assume that, \( G \) is a dss graph such that \( d(G) = 3 \). By our assumption \( d(G) = d(G_{sd ab}) = 3 \), \( G_1 \) and \( G_2 \) are dss graphs. Let \( G_1' = G_1 - \{ u \} \), \( G_2' = G_2 - \{ v \} \). If there is a partition \( d(G_i) = \{ D_1^i, D_2^i, D_3^i \} \), so that \( D_1^i, D_2^i, D_3^i \) are dominating sets for \( G_i \), \( D_1^i, D_2^i, D_3^i \) are dominating sets for \( G_i \) but not for \( G_i' \), then by \( R_6 \) \( d(G) = 4 \), a contradiction to our assumption \( d(G) = 3 \). This implies \( G_1, G_2 \) are graphs which satisfy the condition of the theorem.

Example
\( (G) = 2 \) and \( G \) is dss.

**Proof**

For any \( e \in E(G) \), either \( e \in E(G_1) \) or \( e \in E(G_2) \) or \( e = uv \). If possible assume that \( d(G_{sd}xy) = 3, x \perp y \).

i. \( e = xy \in E(G_1) \).

Proof is similar to the proof of Theorem 12. As in Theorem 12, we get a contradiction, since \( \{ V_1, V_12, V_22 \cup \{ w \} \} \) is a domatic partition for \( G_{1sd}xy \).

ii. \( e = xy \in E(G_2) \).

Contradiction as in (i).

iii. \( e = uv \).

Let \( d(G) = |\{V_1\}, \{V_2\}| \), that satisfies the conditions of R3. Let \( V = X_1 \cup X_2 \), where \( V(X_1) \subseteq V(G_1), V(X_2) \subseteq V(G_2) \). Let \( V_{12} = Y_1 \cup Y_2 \), where \( V(Y_1) \subseteq V(G_1), V(Y_2) \subseteq V(G_2) \). \( V_{22} = Z_1 \cup Z_2 \), where \( V(Z_1) \subseteq V(G_1), V(Z_2) \subseteq V(G_2) \). Let \( Z = \{X_1, Y_1, Z_1\} \). \( Z \) is a domatic partition for \( G_0 \) such that \( X_1, Y_1 \) are dominating sets for \( G_1 \), \( Z_1 \) dominates \( G_1 - \{u\} \), \( d(G) = 1 + \min \{d(G_1), d(G_2)\} = 1 + 2 = 3 \), a contradiction as \( d(G) = 2 \).

From i, ii and iii \( d(G_{sd}uv) \neq 3 \), that is \( d(G_{sd}uv) = 2 \). Hence \( G \) is dss.

**Example**

**Figure 8.** \( d(G_1) = d(G_{1sd}xy) = 2, xy \in E(G_1) \) and \( d(G_2) = d(G_{2sd}xy) = 2, xy \in E(G_2) \). Here \( d(G) = d(G_{sd}ab) = 2 \), for all \( ab \in E(G) \).

**Conclusion**

For the graph \( G, G_1, G_2 \) in the above notations of theorem 13 and 14, we conclude that, if \( G_1 \) and \( G_2 \) are dss, then \( G \) is also dss.

**4. Conclusion and Future Work**

We have used only two graph operations for generating new dss from existing ones. One can generate new dss graphs by applying other graph operations like union, intersection, ring sum etc. Just like the operation to build the larger graphs from smaller ones, one can also do the opposite, decompose a large graph, that is dss graph to smaller graphs that are dss.

**REFERENCES**


