Discrete Time Stability Analysis of a Three Species Eco System

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Abstract In this paper, a three species eco system, involving three pairs is considered to examine the local asymptotic stability: A Prey-Predator, a Commensal-Host and an Ammensal-Enemy. Among the three species, one plays a dual role: A host and an enemy. Time is considered as a discrete unit and the system is modeled as a set of three difference equations. All the equilibrium states are identified and the local asymptotic stability of some of the equilibrium states is examined by considering the perturbation equations. It is observed that among the states, the state in which the Prey and its Host species are washed out(extinct), is spectrally stable and the state where the Predator/Ammensal species is washed out, is asymptotically stable. The results are illustrated with two dimensional plots as well as surface plots.

Keywords Prey, Predator, Ammensal, Commensal, Enemy, Host, Discrete Time, Asymptotic

S_2 depends on S_1 for its survival. S_1 and S_3 form a Commensal – Host pair. That is, S_3 acts as host to S_1 without itself being affected. And S_2 and S_3 form an Ammensal – Enemy pair. That is, S_3 inhibits S_2 without itself being affected. Further the present authors [17] have also carried out a numerical analysis of the same. In the present paper, the three species Ecosystem with time as discrete unit is considered. The equilibrium states are identified and the asymptotic stability of the equilibrium states is examined. A few of them are presented here.

Figure 1. A 3 species ecosystem

1. Introduction

Prey-Predator ecological system was presented by Lotka [1] and Volterra [2] in their classical model. Inspired by this model, several researchers made significant contributions in this area by considering various special types of interactions between the prey and the predator. This has been the motivation for others in bringing a third species into the system thus forming a three species ecological system. More information with examples of all ecological interactions can be found in the book by Paul colinviaux[7]. Recently, Seshagiri rao and Pattabhiramacharyulu[15] worked on three species system by considering interactions like Prey-Predator, Commensal - Host, between the three species, which motivated the present authors[16] to consider a three species Eco system with species S_1, S_2 and S_3 simultaneously having the interactions of Prey-predation, commensalism and ammensalism, as shown in Figure 1, with time as a continuous unit. S_1 and S_2 form a Prey-Predator pair. That is,

2. Notation

N_i : The population of S_i , $i=1, 2, 3$.

a_i : The Natural growth rate of S_i , $i=1, 2, 3$.

a_{ii} : Self inhibition coefficient of S_i , $i=1, 2, 3$. (The rate of decrease of N_i due to insufficient natural resources of S_i)

a_{12} : The rate of decrease of S_1 due to inhibition by S_2 .

a_{13} : The rate of increase of the S_1 due to the promotion by its host S_3 .

a_{21} : The rate of increase of the S_2 due to its attacks on S_1

a_{23} : The rate of decrease of the S_2 due to the harm caused by its enemy S_3 .

$p (= a_{12}/a_{11})$: Coefficient of Prey /Commensal inhibition of

the predator.

$q (= a_{13}/a_{11})$: Coefficient of Commensalism.

$r (= a_{21}/a_{22})$: Coefficient of predator consumption of the prey.

$s (= a_{23}/a_{22})$: Coefficient of Ammensalism.

3. The Model Equations

Based on the interactions of the species S_1 , S_2 and S_3 , the model equations respectively are

$$\begin{aligned}\frac{dN_1}{dt} &= a_1 N_1 - a_{11} N_1^2 + a_{13} N_1 N_3 - a_{12} N_1 N_2 \\ \frac{dN_2}{dt} &= a_2 N_2 - a_{22} N_2^2 + a_{21} N_1 N_2 - a_{23} N_2 N_3 \\ \frac{dN_3}{dt} &= a_3 N_3 - a_{33} N_3^2\end{aligned}$$

Considering time as discrete unit, we have from the model equations

$$\left. \begin{aligned} N_1(t+1) &= \alpha_1 N_1(t) \left[1 - \frac{a_{11}}{\alpha_1} N_1(t) - \frac{a_{12}}{\alpha_1} N_2(t) + \frac{a_{13} N_3(t)}{\alpha_1} \right] \\ N_2(t+1) &= \alpha_2 N_2(t) \left[1 - \frac{a_{22}}{\alpha_2} N_2(t) + \frac{a_{21}}{\alpha_2} N_1(t) - \frac{a_{23}}{\alpha_2} N_3(t) \right] \\ N_3(t+1) &= \alpha_3 N_3(t) \left[1 - \frac{a_{33}}{\alpha_3} N_3(t) \right] \end{aligned} \right\} \quad (3.1)$$

where $1 + a_1 = \alpha_1$, $1 + a_2 = \alpha_2$ and $1 + a_3 = \alpha_3$.

Assuming the notation

$$K_1 = \frac{a_{11}}{\alpha_1}, \quad K_2 = \frac{a_{22}}{\alpha_2}, \quad K_3 = \frac{a_{33}}{\alpha_3}, \quad \beta_{12} = \frac{a_{12}}{\alpha_1},$$

$$\beta_{21} = \frac{a_{21}}{\alpha_2}, \quad \gamma_{13} = \frac{a_{13}}{\alpha_1}, \quad \gamma_{23} = \frac{a_{23}}{\alpha_2},$$

the equations (3.1) become

$$\left. \begin{aligned} N_1(t+1) &= \alpha_1 N_1(t) [1 - K_1 N_1(t) - \beta_{12} N_2(t) + \gamma_{13} N_3(t)] \\ N_2(t+1) &= \alpha_2 N_2(t) [1 - K_2 N_2(t) + \beta_{21} N_1(t) - \gamma_{23} N_3(t)] \\ N_3(t+1) &= \alpha_3 N_3(t) [1 - K_3 N_3(t)] \end{aligned} \right\} \quad (3.2)$$

4. The Equilibrium States

The Equilibrium States of the system are obtained by considering

$$N_i(t+1) = N_i(t), \quad i = 1, 2, 3. \quad (4.1)$$

Solving (4.1), we get eight equilibrium states which can be spread over four distinct classes given by

Fully washed out state:

$$E_1: \bar{N}_1 = 0, \quad \bar{N}_2 = 0, \quad \bar{N}_3 = 0.$$

States in which two species are washed out:

$$E_2: \bar{N}_1 = 0, \quad \bar{N}_2 = 0, \quad \bar{N}_3 = \frac{1}{K_3} \left(1 - \frac{1}{\alpha_3} \right)$$

$$E_3: \bar{N}_1 = 0, \quad \bar{N}_2 = \frac{1}{K_2} \left(1 - \frac{1}{\alpha_2} \right), \quad \bar{N}_3 = 0$$

$$E_4: \bar{N}_1 = \frac{1}{K_1} \left(1 - \frac{1}{\alpha_1} \right), \quad \bar{N}_2 = 0, \quad \bar{N}_3 = 0$$

States in which only one species is washed out :

$$E_5: \bar{N}_1 = 0,$$

$$\bar{N}_2 = \frac{1}{K_2} \left(1 - \frac{1}{\alpha_2} - \frac{\gamma_{23}}{\alpha_3} \left(1 - \frac{1}{\alpha_2} \right) \right),$$

$$\bar{N}_3 = \frac{1}{K_3} \left(1 - \frac{1}{\alpha_3} \right)$$

$$E_6: \bar{N}_1 = \frac{1}{K_1} \left(1 - \frac{1}{\alpha_1} - \frac{\gamma_{13}}{\alpha_3} \left(1 - \frac{1}{\alpha_3} \right) \right),$$

$$\bar{N}_2 = 0,$$

$$\bar{N}_3 = \frac{1}{K_3} \left(1 - \frac{1}{\alpha_3} \right)$$

$$E_7:$$

$$\bar{N}_1 = \frac{\beta_{12} \left(\frac{1}{\alpha_2} - 1 \right) - K_2 \left(\frac{1}{\alpha_2} - 1 \right)}{\beta_{12} \beta_{21} - K_1 K_2},$$

$$\bar{N}_2 = \frac{\beta_{21} \left(\frac{1}{\alpha_1} - 1 \right) - K_1 \left(\frac{1}{\alpha_1} - 1 \right)}{\beta_{12} \beta_{21} - K_2 K_1}, \quad \bar{N}_3 = 0$$

Co- existent state or the Normal steady state:

$$E_8: \bar{N}_1 = \frac{K_2 \left(1 - \frac{1}{\alpha_1} \right) - \beta_{12} \left(1 - \frac{1}{\alpha_2} \right) + \frac{1}{K_3} (\beta_{12} \gamma_{23} + \gamma_{13} K_2) \left(1 - \frac{1}{\alpha_3} \right)}{K_1 K_2 + \beta_{12} \beta_{21}},$$

$$\bar{N}_2 = \frac{\beta_{21} \left(1 - \frac{1}{\alpha_1} \right) + K_1 \left(1 - \frac{1}{\alpha_2} \right) - \frac{1}{K_3} (K_1 \gamma_{23} - \gamma_{13} \beta_{21}) \left(1 - \frac{1}{\alpha_3} \right)}{K_1 K_2 + \beta_{12} \beta_{21}}$$

$$\bar{N}_3 = 0$$

5. Asymptotic Stability of the Equilibrium States

Consider small perturbations $v_i(t)$, $i = 1, 2, 3$ over the equilibrium state. The perturbations are so small that their squares and products are neglected.

That is,

$$N_1 = \bar{N}_1 + v_1(t), \quad N_2 = \bar{N}_2 + v_2(t), \quad N_3 = \bar{N}_3 + v_3(t)$$

Substituting in the difference equations (3.2) we have the equations of perturbations

$$\left. \begin{aligned} v_1(t+1) &\approx \alpha_1 \bar{N}_1 + \alpha_1 v_1 - a_{11} \bar{N}_1^2 - 2a_{11} \bar{N}_1 v_1 - a_{12} \bar{N}_1 \bar{N}_2 - a_{12} \bar{N}_1 v_2 - a_{12} \bar{N}_1 v_1 + a_{13} \bar{N}_1 \bar{N}_3 + a_{13} \bar{N}_1 v_3 \\ &\quad + a_{13} \bar{N}_3 v_1 - \bar{N}_1 \\ v_2(t+1) &\approx \alpha_2 \bar{N}_2 + \alpha_2 v_2 - a_{22} \bar{N}_2^2 - 2a_{22} \bar{N}_2 v_2 - a_{21} \bar{N}_1 \bar{N}_2 + a_{21} \bar{N}_2 v_1 + a_{21} \bar{N}_1 v_2 - a_{23} \bar{N}_2 \bar{N}_3 - a_{23} v_3 \bar{N}_2 \\ &\quad - a_{23} v_2 \bar{N}_3 - \bar{N}_2 \\ v_3(t+1) &\approx \alpha_3 \bar{N}_3 + \alpha_3 v_3 - a_{33} \bar{N}_3^2 - 2a_{33} \bar{N}_3 v_3 - \bar{N}_3 \end{aligned} \right\} \quad (5.1)$$

A. Fully washed out state:

$$5.1. \quad \bar{N}_1(t) = 0, \quad \bar{N}_2(t) = 0, \quad \bar{N}_3(t) = 0$$

Substituting in the perturbation equations (5.1) and neglecting squares and products, we have the perturbation equations for the present state

$$\left. \begin{aligned} v_1(t+1) &= \alpha_1 v_1 [1 - K_1 v_1 - \beta_{12} v_2 + \gamma_{13} v_3] \approx \alpha_1 v_1(t) \\ v_2(t+1) &= \alpha_2 v_2 [1 - K_2 v_2 - \beta_{21} v_1 + \gamma_{23} v_3] \approx \alpha_2 v_2(t) \\ v_3(t+1) &= \alpha_3 v_3 [1 - K_3 v_3] \approx \alpha_3 v_3(t) \end{aligned} \right\} \quad (5.1.1)$$

From the equations (5.1.1) the Jacobian matrix is

$$\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

And the Eigen values are $\lambda_1 = \alpha_1$, $\lambda_2 = \alpha_2$ and $\lambda_3 = \alpha_3$ which are real and positive. For stability analysis, we can consider two cases:

Case 5.1.1: $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_3 > 0$

here, clearly $|\lambda_i| > 1$, $i = 1, 2, 3$.

That is, all the Eigen values lie outside the unit circle. Therefore, the Equilibrium point or fixed point is hyperbolic and is *unstable*. The asymptotic nature of the three variables v_1 , v_2 , v_3 and the surface plot of the trajectories of the perturbed equations when $\alpha_1, \alpha_2, \alpha_3 > 0$ are respectively shown in Fig. 2 and Fig. 3.

Case 5.1.2: $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$

In this case, we have $|\lambda_i| = 1$, $i = 1, 2, 3$.

which means the present state is Spectrally stable or Linearly stable which is shown in Figure 4 and Figure 5.

Figure 2. $a_1 > 0$, $a_2 > 0$, and $a_3 > 0$

Figure 3. $a_1 > 0$, $a_2 > 0$, and $a_3 > 0$

Figure 4. $a_1=0$, $a_2=0$, and $a_3=0$

Figure 5. $a_1=0$, $a_2=0$, and $a_3=0$

B. States in which two species are washed out:

5.2. The State in which S_1 and S_2 Washed out While S_3 is Not

$$\bar{N}_1(t) = 0, \quad \bar{N}_2(t) = 0, \quad \bar{N}_3(t) = \frac{1}{K_3} \left(1 - \frac{1}{\alpha_3} \right)$$

Substitution in the perturbation equations (5.1) result in

$$\left. \begin{aligned} v_1(t+1) &= \frac{a_{33}\alpha_1 + a_{13}(\alpha_3 - 1)}{a_{33}} v_1(t) \\ v_2(t+1) &= \frac{a_{33}\alpha_2 - a_{23}(\alpha_3 - 1)}{a_{33}} v_2(t) \\ v_3(t+1) &= (2 - \alpha_3) v_3(t) \end{aligned} \right\} \quad (5.2.1)$$

From the equations (5.2.1) the Jacobian matrix is

$$\begin{bmatrix} \frac{a_{33}\alpha_1 + a_{13}(\alpha_3 - 1)}{a_{33}} & 0 & 0 \\ 0 & \frac{a_{33}\alpha_2 - a_{23}(\alpha_3 - 1)}{a_{33}} & 0 \\ 0 & 0 & (2 - \alpha_3) \end{bmatrix}$$

And the Eigen values are $\lambda = \frac{a_{33}\alpha_1 + a_{13}(\alpha_3 - 1)}{a_{33}}, \frac{a_{33}\alpha_2 - a_{23}(\alpha_3 - 1)}{a_{33}}$ and $2 - \alpha_3$.

Stability analysis

For this state to be stable, all the Eigen values must be inside the unit circle. That is,

$$\left. \begin{aligned} |\lambda_1| < 1 &\Leftrightarrow 1 < -a_{33}(2 + a_1) < a_{13}(\alpha_3 - 1) \\ &< a_{33}(1 - \alpha_1) \Leftrightarrow -a_{33}(2 + a_1) < a_{13}a_3 < -a_{33}a_1 \\ |\lambda_2| < 1 &\Leftrightarrow a_{33}a_2 < a_{23}a_3 < a_{33}(2 + a_2) \\ |\lambda_3| < 1 &\Leftrightarrow 1 < \alpha_3 < 3 \Leftrightarrow 0 < a_3 < 2 \end{aligned} \right\} \quad (5.2.2)$$

Of the above in (5.2.2.), in the first inequality, the quantity in the middle i.e. $a_{13}a_3$ is positive. Therefore, the condition is invalid. Hence this state is *unstable*. This is diagrammatically shown in Figure 6 and Figure 7.

Figure 6. S_1 and S_2 washed out state

Figure 7. S_1 and S_2 washed out state

5.3. The State in which S_1 and S_3 Washed out But S_2 is Not

$$\bar{N}_1 = 0, \quad \bar{N}_2 = \frac{1}{K_2} \left(1 - \frac{1}{\alpha_2} \right), \quad \bar{N}_3 = 0$$

The perturbation equations for the present state from (5.1) are

$$\left. \begin{aligned} v_1(t+1) &= \left[\alpha_1 - \frac{a_{12}}{a_{22}}(\alpha_2 - 1) \right] v_1(t) \\ v_2(t+1) &= \frac{a_{21}}{a_{22}}(\alpha_2 - 1)v_2(t) + (2 - \alpha_2)v_2(t) - \frac{a_{23}}{a_{22}}(\alpha_2 - 1)v_3(t) \\ v_3(t+1) &= \alpha_3 v_3(t) \end{aligned} \right\} \quad (5.3.1)$$

From the equations (5.3.1) the Jacobian matrix is

$$\begin{bmatrix} \alpha_1 - \frac{a_{12}}{a_{22}}(\alpha_2 - 1) & 0 & 0 \\ \frac{a_{21}}{a_{22}}(\alpha_2 - 1) & 2 - \alpha_2 & -\frac{a_{23}}{a_{22}}(\alpha_2 - 1) \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

And the Eigen values are $\lambda_1 = \alpha_1 - \frac{a_{12}}{a_{22}}(\alpha_2 - 1)$, $\lambda_2 = 2 - \alpha_2$ and $\lambda_3 = \alpha_3$

Stability analysis

The conditions for stability of the equilibrium state are

$$\left. \begin{aligned} |\lambda_1| < 1 &\Leftrightarrow 1 < a_1 a_{22} < a_2 a_{12} < (2 + a_1) a_{22} \\ |\lambda_2| < 1 &\Leftrightarrow 1 < \alpha_2 < 3 \\ |\lambda_3| = 1 &\Leftrightarrow a_3 = 0 \end{aligned} \right\} \dots(5.3.2)$$

In (5.3.2) from the first two inequalities it is clear that for a particular set of values of the parameters, the first two Eigen values lie inside the unit circle. From the third inequality we can conclude that when $a_3=0$, the Eigen value λ_3 lies on the unit circle. Therefore this equilibrium state is Linearly or spectrally stable. The results are shown in Figure 8 and Figure 9.

Figure 8. S1 and S3 washed out state

Figure 9. S1 and S3 washed out state

C. States in which only one species is washed out

5.4. The State in which the Species S_1 is Washed out while S_2 and S_3 are Not

$$\begin{aligned} \bar{N}_1 &= 0, \\ \bar{N}_2 &= \frac{1}{K_2} \left\{ 1 - \frac{1}{\alpha_2} - \frac{\gamma_{23}}{K_3} \left(1 - \frac{1}{\alpha_2} \right) \right\} \\ \bar{N}_3 &= \frac{1}{K_3} \left(1 - \frac{1}{\alpha_3} \right). \end{aligned}$$

Substituting in the perturbation equations (5.1) and neglecting squares and products, we have the perturbation equations for the present state

$$\left. \begin{aligned}
 v_1(t+1) &= \left(\alpha_1 - \frac{a_{12}(a_2 a_{33} - a_3 a_{23})}{a_{22} a_{33}} + \frac{a_1 a_{13}}{a_{33}} \right) v_1(t) \\
 v_2(t+1) &= a_{21} \frac{a_{23}(1-\alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22} a_{33}} v_1(t) + \left(1 - a_2 + \frac{a_3 a_{23}}{a_{33}} \right) v_2(t) + \frac{a_{23}(a_3 a_{23} - a_2 a_{33})}{a_{22} a_{33}} v_3(t) \\
 &+ \alpha_2 \frac{a_{23}(1-\alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22} a_{33}} - a_{22} \left(\frac{a_{23}(1-\alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22} a_{33}} \right)^2 \\
 &- \alpha_2 \gamma_{23} \frac{a_{23}(1-\alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22} a_{33}} \frac{1}{K_3} \frac{\alpha_3 - 1}{\alpha_3} - \frac{a_{23}(1-\alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22} a_{33}} \\
 v_3(t+1) &= (2 - \alpha_3) v_3(t)
 \end{aligned} \right\} \quad (5.4.1)$$

From the above equations the Jacobian matrix is

$$\begin{bmatrix}
 \alpha_1 - \frac{a_{12}(a_2 a_{33} - a_3 a_{23})}{a_{22} a_{33}} + \frac{a_1 a_{13}}{a_{33}} & 0 & 0 \\
 a_{21} \frac{a_{23}(1-\alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22} a_{33}} & 1 - a_2 + \frac{a_3 a_{23}}{a_{33}} & \frac{a_{23}(a_3 a_{23} - a_2 a_{33})}{a_{22} a_{33}} \\
 0 & 0 & 2 - \alpha_3
 \end{bmatrix}$$

And the Eigen values are

$$\lambda_1 = \alpha_1 - \frac{a_{12}(a_2 a_{33} - a_3 a_{23})}{a_{22} a_{33}} + \frac{a_1 a_{13}}{a_{33}}, \quad \lambda_2 = 1 - a_2 + \frac{a_3 a_{23}}{a_{33}} \quad \text{and} \quad \lambda_3 = 2 - \alpha_3$$

Examining the conditions for stability of this equilibrium state, we have

$$\left. \begin{aligned}
 |\lambda_3| < 1 &\Leftrightarrow 1 < \alpha_3 < 3 \\
 |\lambda_2| < 1 &\Leftrightarrow 1 - a_2 + \frac{a_3 a_{23}}{a_{33}} < 1 \\
 |\lambda_1| < 1 &\Leftrightarrow \alpha_1 - \frac{a_{12}(a_2 a_{33} - a_3 a_{23})}{a_{22} a_{33}} < 1
 \end{aligned} \right\} \quad (5.4.2)$$

The inequalities (5.4.2) are not valid because $a_2 a_{33} - a_3 a_{23} > 0$. Therefore, the present state is *unstable*.

The results for a specific case are shown in Figures 10 and Figure 11.

Figure 10. S_1 washed out state

Figure 11. S_1 washed out state

5.5. State in Which the Species S_2 is Washed Out While S_1 and S_3 are Not

$$\bar{N}_1 = \frac{1}{K_1} \left\{ 1 - \frac{1}{\alpha_2} - \frac{\gamma_{13}}{K_3} \left(1 - \frac{1}{\alpha_3} \right) \right\} > 0$$

$$(i.e. a_1 a_{33} + a_3 a_{13} > 0),$$

$$\bar{N}_2 = 0 \quad \text{and}$$

$$\bar{N}_3 = \frac{1}{K_3} \left(1 - \frac{1}{\alpha_3} \right)$$

The perturbation equations for the present state resulting from (5.1) by neglecting squares and products are

$$\left. \begin{aligned} v_1(t+1) &= \alpha_1 (1 + \gamma_{13} \bar{N}_3 - 2K_1 \bar{N}_1) v_1(t) - \alpha_1 \beta_{12} \bar{N}_1 v_2(t) + \alpha_1 \gamma_{13} \bar{N}_1 v_3(t) + \alpha_1 \bar{N}_1 - \alpha_1 K_1 \bar{N}_1^2 + \alpha_1 \gamma_{13} \bar{N}_1 \bar{N}_3 - \bar{N}_1 \\ &= (1 - a_1 - \frac{a_3 a_{13}}{a_{33}}) v_1(t) - \frac{a_{12}}{a_{11}} (a_1 + \frac{a_3 a_{13}}{a_{33}}) v_2(t) + \frac{a_{13}}{a_{11}} (a_1 + \frac{a_3 a_{13}}{a_{33}}) v_3(t) + \alpha_1 \bar{N}_1 - \alpha_1 K_1 \bar{N}_1^2 \\ &\quad + \alpha_1 \gamma_{13} \bar{N}_1 \bar{N}_3 - \bar{N}_1 \\ v_2(t+1) &= \alpha_2 (1 + \beta_{21} \bar{N}_1 - \gamma_{23} \bar{N}_3) v_2(t) \\ &= (\alpha_2 + \frac{a_{21}}{a_{11}} (a_1 + \frac{a_3 a_{13}}{a_{33}}) - \frac{a_3 a_{23}}{a_{33}}) v_2(t) \\ v_3(t+1) &= \alpha_3 (1 - 2K_3 \bar{N}_3) v_3(t) + \alpha_3 \bar{N}_3 - \alpha_3 K_3 \bar{N}_3^2 - \bar{N}_3 \\ &= (2 - \alpha_3) v_3(t) + \alpha_3 \bar{N}_3 - \alpha_3 K_3 \bar{N}_3^2 - \bar{N}_3 \end{aligned} \right\} \quad (5.5.1)$$

From the above equations the Jacobian matrix is

$$\begin{bmatrix} 1 - a_1 - \frac{a_3 a_{13}}{a_{33}} & -\frac{a_{12}}{a_{11}} \left(a_1 + \frac{a_3 a_{13}}{a_{33}} \right) & \frac{a_{13}}{a_{11}} \left(a_1 + \frac{a_3 a_{13}}{a_{33}} \right) \\ 0 & \alpha_2 + \frac{a_{21}}{a_{11}} \left(a_1 + \frac{a_3 a_{13}}{a_{33}} \right) - \frac{a_3 a_{23}}{a_{33}} & 0 \\ 0 & 0 & 2 - \alpha_3 \end{bmatrix}$$

And the Eigen values are

$$\lambda_1 = 1 - a_1 - \frac{a_3 a_{13}}{a_{33}},$$

$$\lambda_2 = \alpha_2 + \frac{a_{21}}{a_{11}} \left(a_1 + \frac{a_3 a_{13}}{a_{33}} \right) - \frac{a_3 a_{23}}{a_{33}}$$

$$\lambda_3 = 2 - \alpha_3$$

Examining the conditions for stability of this equilibrium state, we have

$$\left. \begin{aligned} |\lambda_1| < 1 &\Leftrightarrow a_3 a_{13} < (2 - a_1) a_{33} \text{ and } a_1 > 2 \\ |\lambda_2| < 1 &\Leftrightarrow a_{11} (a_3 a_{23} - (2 + a_2) a_{33}) < a_{21} (a_1 a_{33} + a_3 a_{13}) < a_{11} (a_3 a_{23} - a_2 a_{33}) \\ &\text{and } a_3 a_{23} > a_2 a_{33} \\ |\lambda_3| < 1 &\Leftrightarrow 1 < \alpha_3 < 3 \end{aligned} \right\} \quad (5.5.2)$$

Since all the conditions (inequalities (5.5.2)) above are feasible, for the above conditions the Eigen values lie inside a unit circle. So the equilibrium state is *stable*.

The results are shown in Figure 12 and Figure 13.

Figure 12. S_2 washed out state Figure 13. S_2 washed out state

6. Conclusion

In this paper a three species eco system with various interactions between the species is considered and the asymptotic stability of some of the Equilibrium states is examined. It is observed that among the states, the state in which the Prey and its Host species are washed out (extinct), is spectrally stable and the state where the Predator/Amensal species is washed out, is asymptotically stable. From this it can be understood that in the former case, though the prey and host species are extinct, the Predator survives due to non inhibition by its enemy. While in the later case, the Prey species and its host species continue to exist because neither is dependent on the predator species for its survival. The results are illustrated with two dimensional plots as well as surface plots.

REFERENCES

- [1] Lotka A.J., "Elements of physical Biology, William & Wilking Baltimore, 1925.
- [2] Volterra V., Leconsen La Theorie Mathematique De La Leitte Pou Lavie, Gauthier Villara, Paris, 1931.
- [3] Freedman, H.I., "Deterministic Mathematical Models in population Ecology Marcel-Decker, New York, 1980.
- [4] Kapur J.N., "Mathematical modeling in biology and Medicine", Affiliated east west, 1985.
- [5] Kapur J.N., "Mathematical modeling, Wiley Eastern, 1985.
- [6] Meyer W.J., "Concepts of Mathematical modeling", Mc. Grawhil, 1985.
- [7] Paul Colinvaux, Ecology, John Wiley and Sons Inc., New York, 1986.
- [8] Srinivas N.C., "Some Mathematical aspects of modeling in Bio-medical sciences". PhD thesis, Kakatiya University, 1991.
- [9] Cushing J.M., Integro-Differnetial Equations and Delay Models in Population Dynamics, Lecture Notes in Bio-Mathematics, 20, Springer Verlag, 1997.
- [10] Lakshmi Narayana. K., "A mathematical study of a prey-predator ecological model with a partial cover for the prey and alternative food for the predator", PhD thesis, JNTU, 2005.
- [11] Lakshmi Narayan.K & Pattabhi Ramacharyulu N.Ch, 2007, "A prey predator model with cover for prey and alternate food for the predator and time delay", International Journal of Scientific Computing Vol. 1, pg. 7-14.
- [12] Ravindra Reddy., "A study on mathematical models of Ecological mutualism between two interacting species" PhD thesis, O.U, 2008.
- [13] Archana Reddy R. Pattabhi Ramacharyulu N. Ch & Krishna Gandhi. B., "A stability analysis of two competitive interacting species with harvesting of both the species at a constant rate" International Journal of scientific computing (1); pp 57-68, 2007.
- [14] Bhaskara Rama Sarma & Pattabhi Ramacharyulu N.Ch., "Stability analysis of two species competitive ecosystem". International Journal of logic based intelligent systems, Vol.2 No. 1, 2008.
- [15] Seshagiri Rao.N & Pattabhi Ramacharyulu N.Ch., "Stability of a syn-ecosystem consisting of a Prey -Predator and host commensal to the prey-I" (With mortality rate of prey), 2009.
- [16] D.Ravi kiran, B.Rami reddy, N.Ch.P.Acharyulu, "A Numerical analysis of stability of a three species Eco system consisting of Prey, Predator and a third species which is host to prey and enemy to Predator" ANU Journal of Physical Sciences, Volume 3, No. 1 & 2, 2011.
- [17] D.Ravi kiran, B.Rami reddy, "Stability of a three species ecological system consisting of Prey, predator species and a third species which is a host to the prey and enemy to the predator", Journal of Experimental sciences, Vol.3, No.12, 2012.