Discrete Time Stability Analysis of a Three Species Eco System

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Abstract In this paper, a three species eco system, involving three pairs is considered to examine the local asymptotic stability: A Prey-Predator, a Commensal-Host and an Ammensal-Enemy. Among the three species, one plays a dual role: A host and an enemy. Time is considered as a discrete unit and the system is modeled as a set of three difference equations. All the equilibrium states are identified and the local asymptotic stability of some of the equilibrium states is examined by considering the perturbation equations. It is observed that among the states, the state in which the Prey and its Host species are washed out (extinct), is spectrally stable and the state where the Predator/Ammensal species is washed out, is asymptotically stable. The results are illustrated with two dimensional plots as well as surface plots.

Keywords Prey, Predator, Ammensal, Commensal, Enemy, Host, Discrete Time, Asymptotic

1. Introduction

Prey-Predator ecological system was presented by Lotka [1] and Volterra [2] in their classical model. Inspired by this model, several researchers made significant contributions in this area by considering various special types of interactions between the prey and the predator. This has been the motivation for others in bringing a third species into the system thus forming a three species ecological system. More information with examples of all ecological interactions can be found in the book by Paul colinvaux[7]. Recently, Seshagiri rao and Pattabhiramacharyulu[15] worked on three species system by considering interactions like Prey-Predator, Commensal-Host, between the three species, which motivated the present authors[16] to consider a three species Eco system with species S1, S2 and S3 simultaneously having the interactions of Prey-predation, commensalism and ammensalism, as shown in Figure 1, with time as a continuous unit. S1 and S2 form a Prey-Predator pair. That is, S2 depends on S1 for its survival. S1 and S3 form a Commensal-Host pair. That is, S3 acts as host to S1 without itself being affected. And S2 and S3 form an Ammensal-Enemy pair. That is, S3 inhibits S2 without itself being affected. Further the present authors [17] have also carried out a numerical analysis of the same. In the present paper, the three species Ecosystem with time as discrete unit is considered. The equilibrium states are identified and the asymptotic stability of the equilibrium states is examined. A few of them are presented here.

2. Notation

N_i: The population of S_i, i=1, 2, 3.
a_i: The Natural growth rate of S_i, i=1, 2, 3.
a_{i2}: Self inhibition coefficient of S_i, i=1, 2, 3. (The rate of decrease of N_i due to insufficient natural resources of S_i)
a_{12}: The rate of decrease of S_1 due to inhibition by S_2.
a_{13}: The rate of increase of the S_1 due to the promotion by its host S_3.
a_{21}: The rate of increase of the S_2 due to its attacks on S_1.
a_{23}: The rate of decrease of the S_2 due to the harm caused by its enemy S_3.
P (= a_{12}/a_{11}): Coefficient of Prey/Commensal inhibition of
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the predator.

**Q** (= \(a_{13}/a_{11}\)): Coefficient of Commensalism.

**r** (= \(a_{21}/a_{22}\)): Coefficient of predator consumption of the prey.

**s** (= \(a_{23}/a_{22}\)): Coefficient of Ammensalism.

### 3. The Model Equations

Based on the interactions of the species \(S_1, S_2\) and \(S_3\), the model equations respectively are

\[
\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 + a_{13} N_1 N_3 - a_{12} N_1 N_2
\]

\[
\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{21} N_1 N_2 - a_{23} N_2 N_3
\]

\[
\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2
\]

Considering time as discrete unit, we have from the model equations

\[
N_i(t+1) = \alpha_i N_i(t)
\]

where \(1 + a_1 = \alpha_1\), \(1 + a_2 = \alpha_2\) and \(1 + a_3 = \alpha_3\).

### 4. The Equilibrium States

The Equilibrium States of the system are obtained by considering

\[
N_i(t+1) = N_i(t), \quad i = 1, 2, 3.
\]

Solving (4.1), we get eight equilibrium states which can be spread over four distinct classes given by

**Fully washed out state:**

\[
E_1: \quad \overline{N}_1 = 0, \quad \overline{N}_2 = 0, \quad \overline{N}_3 = 0.
\]

**States in which two species are washed out:**

\[
E_2: \quad \begin{cases}
\overline{N}_1 = 1/K_3 \left(1 - 1/\alpha_3\right) \\
\overline{N}_2 = 1/K_2 \left(1 - 1/\alpha_2\right) \\
\overline{N}_3 = 0
\end{cases}
\]

\[
E_3: \quad \begin{cases}
\overline{N}_1 = 1/K_1 \left(1 - 1/\alpha_1\right) \\
\overline{N}_2 = 0, \\
\overline{N}_3 = 0
\end{cases}
\]

**States in which only one species is washed out:**

\[
E_4: \quad \begin{cases}
\overline{N}_1 = 0 \\
\overline{N}_2 = 1/K_2 \left(1 - 1/\alpha_2\right) \\
\overline{N}_3 = 1/K_3 \left(1 - 1/\alpha_3\right)
\end{cases}
\]

\[
E_5: \quad \begin{cases}
\overline{N}_1 = 1/K_3 \left(1 - 1/\alpha_3\right) \\
\overline{N}_2 = 0, \\
\overline{N}_3 = 1/K_2 \left(1 - 1/\alpha_2\right)
\end{cases}
\]

**States in which only one species is washed out:**

\[
E_6: \quad \begin{cases}
\overline{N}_1 = 1/K_2 \left(1 - 1/\alpha_2\right) \\
\overline{N}_2 = 0, \\
\overline{N}_3 = 1/K_1 \left(1 - 1/\alpha_1\right)
\end{cases}
\]

**States in which only one species is washed out:**

\[
E_7: \quad \begin{cases}
\overline{N}_1 = 1/K_1 \left(1 - 1/\alpha_1\right) \\
\overline{N}_2 = 0, \\
\overline{N}_3 = 1/K_3 \left(1 - 1/\alpha_3\right)
\end{cases}
\]

**States in which only one species is washed out:**

\[
E_8: \quad \begin{cases}
\overline{N}_1 = 1/K_1 \left(1 - 1/\alpha_1\right) \\
\overline{N}_2 = 1/K_3 \left(1 - 1/\alpha_3\right) \\
\overline{N}_3 = 1/K_2 \left(1 - 1/\alpha_2\right)
\end{cases}
\]

\[
\overline{N}_4 = 0
\]

Co- existent state or the Normal steady state:

**E8:**

\[
\overline{N}_1 = \frac{\beta_{12} \left(1 - 1/\alpha_2\right) - K_1 \left(1 - 1/\alpha_1\right)}{\beta_{12} K_1 - K_2 K_3}, \\
\overline{N}_2 = \frac{\beta_{12} \left(1 - 1/\alpha_2\right) - K_1 \left(1 - 1/\alpha_1\right)}{\beta_{12} K_1 - K_2 K_3}, \\
\overline{N}_3 = 0
\]

\[
\overline{N}_4 = \frac{\beta_{12} \left(1 - 1/\alpha_2\right) - K_1 \left(1 - 1/\alpha_1\right)}{\beta_{12} K_1 - K_2 K_3}
\]
5. Asymptotic Stability of the Equilibrium States

Consider small perturbations \( v_i(t), i = 1,2,3 \) over the equilibrium state. The perturbations are so small that their squares and products are neglected. That is,

\[
N_1 = N_1 + v_1(t), \quad N_2 = N_2 + v_2(t), \quad N_3 = N_3 + v_3(t)
\]

Substituting in the difference equations (3.2) we have the equations of perturbations

\[
v_1(t+1) \approx a_1 N_1 + a_1 v_1 - a_{11} N_1 - 2a_{11} N_1 v_1 - a_{12} N_1 N_2 - a_{12} N_1 v_2 - a_{12} N_1 v_3 + a_{13} N_1 N_3 + a_{13} N_1 v_3
\]

\[
+ a_{13} N_3 v_1 - N_1
\]

\[
v_2(t+1) \approx a_2 N_2 + a_2 v_2 - a_{22} N_2 - 2a_{22} N_2 v_2 - a_{21} N_1 N_2 + a_{21} N_1 v_2 + a_{21} N_1 v_1 + a_{23} N_2 N_3 - a_{23} v_3 N_2
\]

\[
- a_{23} v_2 N_3 - N_2
\]

\[
v_3(t+1) \approx a_3 N_3 + a_3 v_3 - a_{33} N_3^2 - 2a_{33} N_3 v_3
\]

(5.1)

A. Fully washed out state:

5.1. \( N_1(t) = 0, \quad N_2(t) = 0, \quad N_3(t) = 0 \)

Substituting in the perturbation equations (5.1) and neglecting squares and products, we have the perturbation equations for the present state

\[
v_1(t+1) = a_1 v_1 \left[ I - K_1 v_1 - \beta_{12} v_2 + \gamma_{13} v_3 \right] \approx a_1 v_1(t)
\]

\[
v_2(t+1) = a_2 v_2 \left[ I - K_2 v_2 - \beta_{21} v_1 + \gamma_{23} v_3 \right] \approx a_2 v_2(t)
\]

\[
v_3(t+1) = a_3 v_3 \left[ I - K_3 v_3 \right] \approx a_3 v_3(t)
\]

(5.1.1)

From the equations (5.1.1) the Jacobian matrix is

\[
\begin{bmatrix}
  a_1 & 0 & 0 \\
  0 & a_2 & 0 \\
  0 & 0 & a_3
\end{bmatrix}
\]

And the Eigen values are \( \lambda_1 = a_1, \lambda_2 = a_2 \) and \( \lambda_3 = a_3 \) which are real and positive. For stability analysis, we can consider two cases:

Case 5.1.1: \( a_1 > 0, \quad a_2 > 0 \) and \( a_3 > 0 \)

here, clearly \( |\lambda_i| > 1, \quad i = 1,2,3 \).

That is, all the Eigen values lie outside the unit circle. Therefore, the Equilibrium point or fixed point is hyperbolic and is unstable. The asymptotic nature of the three variables \( v_1, v_2, v_3 \) and the surface plot of the trajectories of the perturbed equations when \( a_1, a_2, a_3 > 0 \) are respectively shown in Fig. 2 and Fig. 3.

Case 5.1.2: \( a_1 = 0, \quad a_2 = 0 \) and \( a_3 = 0 \)

In this case, we have \( |\lambda_i| = 1, \quad i = 1,2,3 \).

which means the present state is Spectrally stable or Linearly stable which is shown in Figure 4 and Figure 5.
B. States in which two species are washed out:

5.2. The State in which $S_1$ and $S_2$ Washed out While $S_3$ is Not

$$\begin{align*}
\bar{N}_1(t) &= 0, \quad \bar{N}_2(t) = 0, \quad \bar{N}_3(t) = \frac{1}{K_3} \left(1 - \frac{1}{a_3}\right)
\end{align*}$$

Substitution in the perturbation equations (5.1) result in

$$\begin{align*}
v_1(t+1) &= \frac{a_{33}a_1 + a_{33}(a_3 - 1)}{a_{33}} v_1(t) \\
v_2(t+1) &= \frac{a_{33}a_2 - a_{33}(a_3 - 1)}{a_{33}} v_2(t) \\
v_3(t+1) &= (2 - a_3)v_3(t)
\end{align*}$$

(5.2.1)

From the equations (5.2.1) the Jacobian matrix is
Stability analysis

For this state to be stable, all the Eigen values must be inside the unit circle. That is,
\[
\begin{align*}
|\lambda_1| < 1 & \iff -a_{33}(2 + a_1) < a_{13}(a_3 - 1) \\
& < a_{33}(1 - a_1) \iff -a_{33}(2 + a_1) < a_{13}a_3 < -a_{33}a_1
\end{align*}
\]
\[
|\lambda_2| < 1 \iff a_{33}a_2 < a_{23}a_3 < a_{33}(2 + a_2)
\]
\[
|\lambda_3| < 1 \iff 1 < a_3 < 3 \iff 0 < a_3 < 2
\]

Of the above in (5.2.2.), in the first inequality, the quantity in the middle i.e. $a_{13}a_3$ is positive. Therefore, the condition is invalid. Hence this state is unstable. This is diagrammatically shown in Figure 6 and Figure 7.

5.3. The State in which $S_1$ and $S_3$ Washed out But $S_2$ is Not

\[
\begin{align*}
\overline{N}_1 = 0, \quad \overline{N}_2 &= \frac{1}{\overline{K}_2} \left(1 - \frac{1}{\alpha_2}\right), \quad \overline{N}_3 = 0
\end{align*}
\]

The perturbation equations for the present state from (5.1) are
\[
\begin{align*}
v_1(t+1) &= \left[a_1 - \frac{a_{12}}{a_{22}}(a_2 - 1)\right]v_1(t) \\
v_2(t+1) &= \frac{a_{21}}{a_{22}}(a_2 - 1)v_2(t) + (2 - a_2)v_2(t) - \frac{a_{23}}{a_{22}}(a_2 - 1)v_3(t) \\
v_3(t+1) &= a_3v_3(t)
\end{align*}
\]
\[
(5.3.1)
\]
From the equations (5.3.1) the Jacobian matrix is

\[
\begin{bmatrix}
\alpha_1 - \frac{a_{12}}{a_{22}} (\alpha_2 - 1) & 0 & 0 \\
\frac{a_{21}}{a_{22}} (\alpha_2 - 1) & 2 - \alpha_2 - \frac{a_{23}}{a_{22}} (\alpha_2 - 1) & 0 \\
0 & 0 & \alpha_3
\end{bmatrix}
\]

And the Eigen values are \( \lambda_1 = \alpha_1 - \frac{a_{12}}{a_{22}} (\alpha_2 - 1) \), \( \lambda_2 = 2 - \alpha_2 \) and \( \lambda_3 = \alpha_3 \)

**Stability analysis**

The conditions for stability of the equilibrium state are

\[
\begin{align*}
|\lambda_1| < 1 & \iff 1 < a_1 a_{22} < a_2 a_{12} < (2 + a_1) a_{22} \\
|\lambda_2| < 1 & \iff 1 < \alpha_2 < 3 \\
|\lambda_3| = 1 & \iff a_3 = 0
\end{align*}
\]

In (5.3.2) from the first two inequalities it is clear that for a particular set of values of the parameters, the first two Eigen values lie inside the unit circle. From the third inequality we can conclude that when \( a_3 = 0 \), the Eigen value \( \lambda_3 \) lies on the unit circle. Therefore this equilibrium state is Linearly or spectrally stable. The results are shown in Figure 8 and Figure 9.

![Figure 8. S1 and S3 washed out state](image1)

![Figure 9. S1 and S3 washed out state](image2)

C. States in which only one species is washed out

5.4. The State in which the Species S1 is Washed out while S2 and S3 are Not

\[
\begin{align*}
\overline{N}_1 &= 0, \\
\overline{N}_2 &= \frac{1}{K_2} \left\{ 1 - \frac{1}{\alpha_2} \frac{\gamma_{23}}{K_3} \left( 1 - \frac{1}{\alpha_2} \right) \right\}, \\
\overline{N}_3 &= \frac{1}{K_3} \left\{ 1 - \frac{1}{\alpha_3} \right\}
\end{align*}
\]

Substituting in the perturbation equations (5.1) and neglecting squares and products, we have the perturbation equations for the present state
\[ v_1(t+1) = \left( \alpha_1 - \frac{a_{12}(a_{23}a_{33} - a_{32}a_{23})}{a_{22}a_{33}} + \frac{a_{13}a_{13}}{a_{33}} \right) v_1(t) \]

\[ v_2(t+1) = a_{21} \frac{a_{23}(1 - \alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22}a_{33}} - v_1(t) + \left( 1 - a_2 + \frac{a_{32}a_{33}}{a_{33}} \right) v_2(t) + \frac{a_{23}(a_{23}a_{33} - a_{32}a_{23})}{a_{22}a_{33}} v_3(t) \]

\[ + \alpha_2 \frac{a_{23}(1 - \alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22}a_{33}} - a_{22} \left( a_{23}(1 - \alpha_3) + a_{33}(\alpha_2 - 1) \right)^2 \]

\[ - a_{22} \gamma_{23} \frac{a_{23}(1 - \alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22}a_{33}} \]

\[ \alpha_3 - 1 \]

\[ \frac{a_{23}(1 - \alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22}a_{33}} \]

\[ K_3 \alpha_3 \]

\[ - a_{23}(1 - \alpha_3) + a_{33}(\alpha_2 - 1) \]

\[ \frac{a_{23}(1 - \alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22}a_{33}} \]

\[ v_3(t+1) = (2 - \alpha_3)v_3(t) \]

From the above equations the Jacobian matrix is

\[
\begin{pmatrix}
\alpha_1 - \frac{a_{12}(a_{23}a_{33} - a_{32}a_{23})}{a_{22}a_{33}} + \frac{a_{13}a_{13}}{a_{33}} & 0 & 0 \\
\frac{a_{21}(1 - \alpha_3) + a_{31}(\alpha_2 - 1)}{a_{22}a_{33}} & 1 - a_2 + \frac{a_{32}a_{33}}{a_{33}} & a_{23}(a_{23}a_{33} - a_{32}a_{23}) \\
0 & \frac{a_{23}(1 - \alpha_3) + a_{33}(\alpha_2 - 1)}{a_{22}a_{33}} & 2 - a_3
\end{pmatrix}
\]

And the Eigen values are

\[ \lambda_1 = \alpha_1 - \frac{a_{12}(a_{23}a_{33} - a_{32}a_{23})}{a_{22}a_{33}} + \frac{a_{13}a_{13}}{a_{33}} \]

\[ \lambda_2 = 1 - a_2 + \frac{a_{32}a_{33}}{a_{33}} \]

\[ \lambda_3 = 2 - a_3 \]

Examining the conditions for stability of this equilibrium state, we have

\[
\begin{align*}
|\lambda_1| < 1 & \iff 1 < \alpha_3 < 3 \\
|\lambda_2| < 1 & \iff 1 - a_2 + \frac{a_{32}a_{33}}{a_{33}} < 1 \\
|\lambda_3| < 1 & \iff \alpha_1 - \frac{a_{12}(a_{23}a_{33} - a_{32}a_{23})}{a_{22}a_{33}} < 1
\end{align*}
\]

The inequalities (5.4.2) are not valid because \(a_{23}a_{33} - a_{32}a_{23}\geq0\). Therefore, the present state is \textit{unstable}. The results for a specific case are shown in Figures 10 and Figure 11.

![Figure 10](image1.png) S₁ washed out state

![Figure 11](image2.png) S₁ washed out state
5.5. State in Which the Species S₂ is Washed Out While S₁ and S₃ are Not

\[
\bar{N}_1 = \frac{1}{K_1} \left\{ 1 - \frac{1}{a_2} - \frac{\gamma_{13}}{K_3} \left( 1 - \frac{1}{a_3} \right) \right\} > 0
\]

(i.e. \( a_1a_{33} + a_3a_{13} > 0 \)),

\[
\bar{N}_2 = 0 \quad \text{and} \quad \bar{N}_3 = \frac{1}{K_3} \left( 1 - \frac{1}{a_3} \right)
\]

The perturbation equations for the present state resulting from (5.1) by neglecting squares and products are

\[
v_1(t + 1) = a_1(1 + \gamma_{13} \bar{N}_3 - 2K_1 \bar{N}_1) v_1(t) - a_1 \beta_{12} \bar{N}_1 v_2(t) + a_1 \gamma_{13} \bar{N}_1 v_3(t) + a_1 \bar{N}_1 - a_1 K_1 \bar{N}_1^2 + a_1 \gamma_{13} \bar{N}_1 \bar{N}_3 - \bar{N}_1
\]

\[
v_2(t + 1) = a_2(1 + \beta_{21} \bar{N}_1 - \gamma_{23} \bar{N}_3) v_2(t)
\]

\[
v_3(t + 1) = a_3(1 - 2K_3 \bar{N}_3) v_3(t) + a_3 \bar{N}_3 - a_3 K_3 \bar{N}_3^2 - \bar{N}_3
\]

From the above equations the Jacobian matrix is

\[
\begin{bmatrix}
1-a_1-rac{a_3a_{13}}{a_{33}} & -rac{a_{12}}{a_{11}}(a_1 + \frac{a_3a_{13}}{a_{33}}) & \frac{a_{13}}{a_{11}}(a_1 + \frac{a_3a_{13}}{a_{33}}) \\
0 & a_2 + \frac{a_{21}}{a_{11}}(a_1 + \frac{a_3a_{13}}{a_{33}}) & -\frac{a_2a_{33}}{a_{33}} \\
0 & 0 & 2-a_3
\end{bmatrix}
\]

And the Eigen values are

\[
\lambda_1 = 1 - a_1 - \frac{a_3a_{13}}{a_{33}},
\]

\[
\lambda_2 = a_2 + \frac{a_{21}}{a_{11}}(a_1 + \frac{a_3a_{13}}{a_{33}}) - \frac{a_2a_{33}}{a_{33}}
\]

\[
\lambda_3 = 2 - a_3
\]

Examining the conditions for stability of this equilibrium state, we have

\[
\begin{align*}
|\lambda_1| < 1 & \iff a_3a_{13} < (2 - a_1)a_{33} \quad \text{and} \quad a_1 > 2 \\
|\lambda_2| < 1 & \iff a_{11}(a_3a_{23} - (2 + a_2)a_{33}) < a_{21}(a_1a_{33} + a_3a_{13}) < a_{11}(a_3a_{23} - a_2a_{33}) \\
\text{and} \quad a_1a_{23} & > a_2a_{33} \\
|\lambda_3| < 1 & \iff 1 < a_3 < 3
\end{align*}
\]

Since all the conditions (inequalities (5.5.2)) above are feasible, for the above conditions the Eigen values lie inside a unit circle. So the equilibrium state is stable.

The results are shown in Figure 12 and Figure 13.
6. Conclusion

In this paper a three species eco system with various interactions between the species is considered and the asymptotic stability of some of the Equilibrium states is examined. It is observed that among the states, the state in which the Prey and its Host species are washed out (extinct), is spectrally stable and the state where the Predator/Ammensal species is washed out, is asymptotically stable. From this it can be understood that in the former case, though the prey and host species are extinct, the Predator survives due to non inhibition by its enemy. While in the later case, the Prey species and its host species continue to exist because neither is dependent on the predator species for its survival. The results are illustrated with two dimensional plots as well as surface plots.

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