Solving Some Definite Integrals by Using Maple

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Abstract This study uses the mathematical software Maple for the auxiliary tool to evaluate two types of definite integrals. We can obtain the closed forms of these two types of definite integrals by using differentiation with respect to a parameter and Leibniz differential rule. At the same time, we provide some definite integrals to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple.

Keywords Definite Integrals, Closed Forms, Differentiation With Respect to a Parameter, Leibniz Differential Rule, Maple

1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of

Maple, [1-7] can be adopted as references.

In calculus and engineering mathematics courses, we learnt many methods to solve the integral problems, including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, and so on. In this paper, we study the evaluation of the following two types of definite integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_0^{\pi} \frac{\cos^m x}{(a + b \cos x)^{n+m+1}} dx \quad (1)$$

$$\int_0^{\pi/2} \frac{\cos^{2p} x \cdot \sin^{2q} x}{(r + \lambda \cos^2 x + \beta \sin^2 x)^{n+p+q+1}} dx \quad (2)$$

where m, n, p, q are non-negative integers, a, b, r, λ, β are real numbers, and $a > |b|$, $2r + \lambda + \beta > |\lambda - \beta|$. We can obtain the closed forms of these two types of definite integrals by using differentiation with respect to a parameter and Leibniz differential rule; these are the main results of this paper (i.e., Theorems 1, 2). The study of related integral problems can refer to [8-26]. Simultaneously, we can compare the method used in this article with that in [27]. On the other hand, we provide some definite integrals to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce some notations and a formula used in this study.

2.1. Notations

- (i) the P -th order derivative of the function $u(x)$

is denoted by $u^{(p)}(x)$, where p is a non-negative integer.

(ii) Suppose s, t are real numbers, we define

$$(s)_t = s(s-1)\cdots(s-t+1), \text{ and } (s)_0 = 1.$$

(iii) Suppose m, n are positive integers, we define

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}, \text{ and } \binom{m}{0} = 1.$$

2.2. Integral Formula ([28])

Suppose a, b are real numbers and $a > |b|$, then the definite integral $\int_0^\pi \frac{1}{a+b\cos x} dx = \frac{\pi}{\sqrt{a^2-b^2}}$.

Next, we introduce two important theorems used in this paper.

2.3. Differentiation With Respect to a Parameter ([29])

Suppose c, d, λ, β are real numbers and the function $f(a, x)$ is defined on $[c, d] \times [\lambda, \beta]$. If $f(a, x)$ and its partial derivative $\frac{\partial f}{\partial a}(a, x)$ are continuous functions on

$[c, d] \times [\lambda, \beta]$. Then $F(a) = \int_\lambda^\beta f(a, x) dx$ is differentiable

on the open interval (c, d) , and its derivative $\frac{d}{da} F(a) =$

$$\int_\lambda^\beta \frac{\partial f}{\partial a}(a, x) dx \text{ for all } a \in (c, d).$$

2.4. Leibniz Differential Rule ([30])

Suppose n is an non-negative integer, $f(x)$ and $g(x)$ are n -times differentiable functions. Then the n -th order derivative of the product function $f(x) \cdot g(x)$,

$$(f \cdot g)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}$$

The following is the first result of this study, we obtain the closed form of definite integral (1).

2.5. Theorem 1

Assume m, n are non-negative integers, a, b are real numbers, and $a > |b|$. Then the definite integral

$$\begin{aligned} & \int_0^\pi \frac{\cos^m x}{(a+b\cos x)^{n+m+1}} dx \\ &= \frac{(-1)^n m! n! \pi}{(n+m)!} \times \\ & \sum_{k=0}^n \sum_{j=0}^m \frac{\left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k} \left(-\frac{1}{2}-k\right)_j \left(-\frac{1}{2}-n+k\right)_{m-j}}{k!(n-k)!j!(m-j)!} \times \end{aligned}$$

$$(-1)^j \cdot (a+b)^{-1/2-k-j} \cdot (a-b)^{-1/2-n+k-m+j} \quad (3)$$

2.5.1. Proof

By the integral formula, we obtain

$$\begin{aligned} & \int_0^\pi \frac{1}{a+b\cos x} dx \\ &= \frac{\pi}{\sqrt{a^2-b^2}} \\ &= \pi \cdot (a+b)^{-1/2} (a-b)^{-1/2} \end{aligned} \quad (4)$$

Using differentiation with respect to a parameter and Leibniz differential rule, differentiating n -times with respect to a on both sides of (4), we have

$$\begin{aligned} & \int_0^\pi \frac{(-1)^n n!}{(a+b\cos x)^{n+1}} dx \\ &= \pi \cdot \sum_{k=0}^n \binom{n}{k} [(a+b)^{-1/2}]^{(k)} [(a-b)^{-1/2}]^{(n-k)} \\ &= \pi \cdot \sum_{k=0}^n \binom{n}{k} \left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k} (a+b)^{-1/2-k} (a-b)^{-1/2-n+k} \end{aligned} \quad (5)$$

Therefore,

$$\begin{aligned} & \int_0^\pi \frac{1}{(a+b\cos x)^{n+1}} dx \\ &= \frac{\pi}{(-1)^n n!} \cdot \sum_{k=0}^n \binom{n}{k} \left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k} (a+b)^{-1/2-k} (a-b)^{-1/2-n+k} \end{aligned} \quad (6)$$

Also, by differentiation with respect to a parameter and Leibniz differential rule, differentiating m -times with respect to b on both sides of (6), we obtain

$$\begin{aligned} & (-1)^m \frac{(n+m)!}{n!} \cdot \int_0^\pi \frac{\cos^m x}{(a+b\cos x)^{n+m+1}} dx \\ &= \frac{\pi}{(-1)^n n!} \times \end{aligned}$$

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k} \sum_{j=0}^m \binom{m}{j} [(a+b)^{-1/2-k}]^{(j)} \times \\ & [(a-b)^{-1/2-n+k}]^{(m-j)} \\ &= \frac{(-1)^n \pi}{n!} \times \end{aligned}$$

$$\begin{aligned} & \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} \left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k} \left(-\frac{1}{2}-k\right)_j \left(-\frac{1}{2}-n+k\right)_{m-j} \times \\ & (-1)^{m-j} \cdot (a+b)^{-1/2-k-j} \cdot (a-b)^{-1/2-n+k-m+j} \end{aligned} \quad (7)$$

Thus,

$$\begin{aligned} & \int_0^\pi \frac{\cos^m x}{(a+b\cos x)^{n+m+1}} dx \\ &= \frac{(-1)^n m! n! \pi}{(n+m)!} \times \\ & \sum_{k=0}^n \sum_{j=0}^m \frac{\left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k} \left(-\frac{1}{2}-k\right)_j \left(-\frac{1}{2}-n+k\right)_{m-j}}{k!(n-k)!j!(m-j)!} \times \\ & (-1)^j \cdot (a+b)^{-1/2-k-j} \cdot (a-b)^{-1/2-n+k-m+j} \blacksquare \end{aligned}$$

Next, we determine the closed form of definite integral (2).

2.6. Theorem 2

If n, p, q are non-negative integers, r, λ, β are real numbers, and $2r + \lambda + \beta > |\lambda - \beta|$. Then the definite integral

$$\begin{aligned} & \int_0^{\pi/2} \frac{\cos^{2p} x \cdot \sin^{2q} x}{(r + \lambda \cos^2 x + \beta \sin^2 x)^{n+p+q+1}} dx \\ &= \frac{(-2)^{n+p+q} \cdot n! \pi}{(n+p+q)!} \times \\ & \sum_{k=0}^n \frac{\left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k} \left(-\frac{1}{2}-k\right)_p \left(-\frac{1}{2}-n+k\right)_q}{k!(n-k)!} \times \\ & (2r+2\lambda)^{-1/2-k-p} \cdot (2r+2\beta)^{-1/2-n+k-q} \quad (8) \end{aligned}$$

2.6.1. Proof

Because

$$\begin{aligned} & \int_0^{\pi/2} \frac{1}{(r + \lambda \cos^2 x + \beta \sin^2 x)^{n+1}} dx \\ &= \int_0^{\pi/2} \frac{1}{\left(r + \lambda \cdot \frac{1 + \cos 2x}{2} + \beta \cdot \frac{1 - \cos 2x}{2}\right)^{n+1}} dx \\ &= \int_0^{\pi/2} \frac{2^{n+1}}{[(2r + \lambda + \beta) + (\lambda - \beta) \cos 2x]^{n+1}} dx \\ &= 2^n \cdot \int_0^\pi \frac{1}{[(2r + \lambda + \beta) + (\lambda - \beta) \cos x]^{n+1}} dx \\ &= \frac{2^n \pi}{(-1)^n n!} \cdot \sum_{k=0}^n \binom{n}{k} \left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k} (2r+2\lambda)^{-1/2-k} (2r+2\beta)^{-1/2-n+k} \\ & \quad \text{(by (6))} \\ &= \frac{2^n \pi}{(-1)^n} \cdot \sum_{k=0}^n \frac{\left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k}}{k!(n-k)!} (2r+2\lambda)^{-1/2-k} (2r+2\beta)^{-1/2-n+k} \quad (9) \end{aligned}$$

Also, using differentiation with respect to a parameter and Leibniz differential rule, differentiating P -times with respect to λ on both sides of (9), we obtain

$$\begin{aligned} & \int_0^{\pi/2} \frac{\cos^{2p} x}{(r + \lambda \cos^2 x + \beta \sin^2 x)^{n+p+1}} dx \\ &= \frac{(-2)^{n+p} \cdot n! \pi}{(n+p)!} \cdot \sum_{k=0}^n \frac{\left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k} \left(-\frac{1}{2}-k\right)_p}{k!(n-k)!} \times \\ & (2r+2\lambda)^{-1/2-k-p} \cdot (2r+2\beta)^{-1/2-n+k} \quad (10) \end{aligned}$$

Again, by differentiation with respect to a parameter and Leibniz differential rule, differentiating q -times with respect to β on both sides of (10), then

$$\begin{aligned} & \int_0^{\pi/2} \frac{\cos^{2p} x \cdot \sin^{2q} x}{(r + \lambda \cos^2 x + \beta \sin^2 x)^{n+p+q+1}} dx \\ &= \frac{(-2)^{n+p+q} \cdot n! \pi}{(n+p+q)!} \times \\ & \sum_{k=0}^n \frac{\left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{n-k} \left(-\frac{1}{2}-k\right)_p \left(-\frac{1}{2}-n+k\right)_q}{k!(n-k)!} \times \\ & (2r+2\lambda)^{-1/2-k-p} \cdot (2r+2\beta)^{-1/2-n+k-q} \quad \blacksquare \end{aligned}$$

3. Examples

In the following, for the two types of definite integrals in this study, we propose some examples and use Theorems 1 and 2 to determine their closed forms. On the other hand, we employ Maple to calculate the approximations of these definite integrals and their closed forms for verifying our answers.

3.1. Example 1

In Theorem 1, taking $m = 4, n = 2, a = 5, b = 3$, we obtain the following definite integral

$$\begin{aligned} & \int_0^\pi \frac{\cos^4 x}{(5 + 3 \cos x)^7} dx \\ &= \frac{\pi}{15} \cdot \sum_{k=0}^2 \sum_{j=0}^4 \frac{\left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{2-k} \left(-\frac{1}{2}-k\right)_j \left(-\frac{5}{2}+k\right)_{4-j}}{k!(2-k)!j!(4-j)!} \times \\ & (-1)^j \cdot 8^{-1/2-k-j} \cdot 2^{-13/2+k+j} \quad (11) \end{aligned}$$

Using Maple to verify the correctness of (11).

```
>evalf(int((cos(x))^4/(5+3*cos(x))^7,x=0..Pi),14);
```

0.0026408801975462

```
>evalf(Pi/15*sum(sum(product(-1/2-r,r=0..(k-1))*product(-1/2-s,s=0..(1-k))*product(-1/2-k,t=0..(j-1))*product(-5/2+k-w,w=0..(3-j))/(k!*(2-k)!*(j!*(4-j)!*(-1)^j*8^(-1/2-k-j)*2^(-13/2+k+j)),k=0..2),j=0..4),14);
```

0.0026408801975462

3.2. Example 2

In Theorem 1, let $m = 5, n = 3, a = 7, b = -2$, then the definite integral

$$\int_0^\pi \frac{\cos^5 x}{(7 - 2 \cos x)^9} dx$$

$$= \frac{-\pi}{56} \cdot \sum_{k=0}^3 \sum_{j=0}^5 \frac{\left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{3-k} \left(-\frac{1}{2}-k\right)_j \left(-\frac{7}{2}+k\right)_{5-j}}{k!(3-k)!j!(5-j)!} \times$$

$$(-1)^j \cdot 5^{-1/2-k-j} \cdot 9^{-17/2+k+j} \quad (12)$$

Next, we use Maple to verify the correctness of (12).

```
>evalf(int((cos(x))^5/(7-2*cos(x))^9,x=0..Pi),14);
```

2.1552539566445·10⁻⁷

```
>evalf(-Pi/56*sum(sum(product(-1/2-r,r=0..(k-1))*product(-1/2-s,s=0..(2-k))*product(-1/2-k-t,t=0..(j-1))*product(-7/2+k-w,w=0..(4-j))/(k!*(3-k)!*j!*(5-j)!)*(-1)^j*5^(-1/2-k-j)*9^(-17/2+k+j),k=0..3),j=0..5),14);
```

2.1552539566444·10⁻⁷

3.3. Example 3

In Theorem 2, let $n = 4, p = 2, q = 1, r = 2, \lambda = 3, \beta = 2$, we can determine the definite integral

$$\int_0^{\pi/2} \frac{\cos^4 x \cdot \sin^2 x}{(2 + 3 \cos^2 x + 2 \sin^2 x)^8} dx$$

$$= \frac{-64\pi}{105} \cdot \sum_{k=0}^4 \frac{\left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{4-k} \left(-\frac{1}{2}-k\right)_2 \left(-\frac{9}{2}+k\right)_1}{k!(4-k)!} \cdot 10^{-5/2-k} \cdot 8^{-11/2+k} \quad (13)$$

```
>evalf(int((((cos(x))^4*(sin(x))^2)/(2+3*(cos(x))^2+2*(sin(x))^2))^8,x=0..Pi/2),14);
```

5.1015015791996·10⁻⁷

```
>evalf(-64*Pi/105*sum(product(-1/2-r,r=0..(k-1))*product(-1/2-s,s=0..(3-k))*product(-1/2-k-t,t=0..1)*(-9/2+k)/(k!*(4-k)!)*10^(-5/2-k)*8^(-11/2+k),k=0..4),14);
```

5.1015015791995·10⁻⁷

3.4. Example 4

In Theorem 2, if $n = 3, p = 2, q = 4, r = 5, \lambda = -2, \beta = 4$, then the definite integral

$$\int_0^{\pi/2} \frac{\cos^4 x \cdot \sin^8 x}{(5 - 2 \cos^2 x + 4 \sin^2 x)^{10}} dx$$

$$= \frac{-8\pi}{945} \cdot \sum_{k=0}^3 \frac{\left(-\frac{1}{2}\right)_k \left(-\frac{1}{2}\right)_{3-k} \left(-\frac{1}{2}-k\right)_2 \left(-\frac{7}{2}+k\right)_4}{k!(3-k)!} \cdot 6^{-5/2-k} \cdot 18^{-15/2+k} \quad (14)$$

```
>evalf(int((((cos(x))^4*(sin(x))^8)/(5-2*(cos(x))^2+4*(sin(x))^2))^10,x=0..Pi/2),14);
```

2.7089130065827·10⁻¹⁰

```
>evalf(-8*Pi/945*sum(product(-1/2-r,r=0..(k-1))*product(-1/2-s,s=0..(2-k))*product(-1/2-k-t,t=0..1)*product(-7/2+k-w,w=0..3)/(k!*(3-k)!*6^(-5/2-k)*18^(-15/2+k),k=0..3),14);
```

2.7089130065828·10⁻¹⁰

4. Conclusion

In this paper, we provide a new technique to solve two types of definite integrals, and we know that differentiation with respect to a parameter and Leibniz differential rule play significant roles in the theoretical inferences of this study. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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