A New Inversion Formula for Laplace Transforms and the Notion of Evenness

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Abstract We prove a new inversion formula for Laplace transform in form $\text{SiCoLL}(S(x)) = c S(x), c = \text{const.}$, where Si is the sine-transform, Co is the cosine-transform of Fourier for not negative $x$, and L is the Laplace transform on real axis for not negative $x$.

Keywords Laplace Transform, New Inversion Formula for Laplace Transform, Evenness for Laplace Transform, Fourier Transform

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1 Introduction

A new inversion formula for Laplace transforms is considered. In the formula we use only the positive values of variables $x \in [0, +\infty)$.

In the old famous inversion formula (the work of Davis ([1])) we use the analytic continuation of transforms from $[0, \infty)$ to the axis $(a - i\infty, a + i\infty)$. The continuation is not required in the inversion formula of theorem 2 (the second part of theorem 2). The decision of the task is led in the terms of repeated definite integrals on $[0, \infty)$ (the second part of theorem 2 and example 1).

An equality

$$\text{LL}_s(t)(\cdot)(x) = \text{SiCoLL}_s(t)(\cdot)(x), x \in (0, \infty),$$

is one of the main results of the article (the theorem 2), where

$$\text{LL}_s(t)(\cdot)(x) = \int_0^\infty e^{-xt} S_1(t) dt, x \in [0, \infty),$$

and a definitions of $S(x)$ functions are considered in theorem 2.

The main lemma 2 is the basis of proof of theorem 2 and the basis of verification of too general conditions of theorem 1. From lemma 2 we obtain the new classes of the Laplace transforms $f(p) : f(-p) = f(p)$.

In lemma 1 the following fact is proved: the analytical continuation of the double Laplace transform $\text{LL}_s(\sqrt{t})(p) = \text{LL} \downarrow \text{s}(\sqrt{t})(\cdot)(p)$ round 0 clockwise is locked on the axis $(0, \infty)$ with

$$\text{LL}_u \text{s}(\sqrt{t})(\cdot)(p),$$

where $\text{LL}_u \text{s}(\sqrt{t})(\cdot)(p)$ is the analytical continuation of $\text{LL}_s(\sqrt{t})(\cdot)(p)$ round 0 anticlockwise.

The new inversion formula for Laplace transforms is formulated in the theorem 2:

$$\text{CoSiLL}_s(t)(\cdot)(x) = (\pi / 2)^2 S_0(t)(\cdot)(x), x \in (0, \infty),$$

if

$$S_0(t) = \text{Si}(S(t))(\cdot)(x), S(p) \in G_1(\star)$$

(the first part of theorem 2, a definition of the $G_1(\star)$ class is in the second section of article), or (the second part of theorem 2)

$$\text{CoSiLL}_s(t)(\cdot)(x) = (\pi / 2)^2 S_1(t)(\cdot)(x), x \in (0, \infty),$$

for the function $S_1(x)$

$$S_1(-a) = S_1(a) = S_1(0) = 0, S_1(-x) = -S_1(x), x \in [0, \infty),$$

$$S_1(x) = 0, |a| > 0,$$

for a constant $a \in (0, \infty)$ ($dS_1^2(t)/dt \text{ is continuous on } [-a, a]$).

The traditional inversion formula for Laplace transform is considered, for instance, in works of Davis ([1])

2 The new inversion formula for Laplace transform.

We will use a designations
\[ LF_+ S(t)(\cdot)(\lambda) = \int_0^\infty e^{-\lambda x} dx \int_{-\infty}^\infty e^{zit} S(t) dt, \lambda \in [0, \infty); \]

\[ F_\pm S(t)(\cdot)(x) = \int_{-\infty}^\infty e^{\pm zit} S(t) dt, x \in (-\infty, \infty), \]

\[ F_\pm^0 S(t)(\cdot)(x) = \int_0^\infty e^{\pm zit} S(t) dt, x \in (-\infty, \infty). \]

**Theorem 1.**

\[ LLS_0(t)(\cdot)(x) = SiCoS_0(t)(\cdot)(x), x \in (0, \infty), \]

if

1. \(S_0(-t) = -S_0(t), S(0) = 0, \)

\[ R(ix) = R(-ix), x \in [0, +\infty), \]

where

\[ R(p) = \int_0^\infty e^{-px} S_0(x) dx, \ Re p \geq 0; \]

2. and, if

\[ \int_0^\infty |S_0(x)| dx < \infty, \]

\[ \int_0^\infty |d^2S_0(x)/dx^2| dx < \infty, \]

\[ \int_0^\infty |R(x)| dx < \infty. \]

**Proof.**

Lets \( I_{+}(t) = 1, t \geq 0; I_{-}(t) = -1, t < 0. \)

For \( S_0(-t) = -S_0(t), \)

\[ \int_{-\infty}^0 e^{uit}(-S_0(t)) dt = \int_{-\infty}^\infty e^{-uit}S_0(t) dt = R(-iu) = R(iu), u \in [0, \infty), \]

and we obtain

\[ Im F_+^0 F_+ (I_{+}(t)S_0(t))(-u) = 2Im R(-iu) = 2\int_{-\infty}^\infty e^{zu} dx \int_0^\infty e^{zit} S_0(t) dt, u \in [0, \infty). \]

Obviously ( after changing of limits of integration )

\[ \int_0^\infty e^{izu} dx \int_{-\infty}^\infty e^{zit} S_0(t) dt = iLLS_0(t)(\cdot)(u), \]

\( u \in [0, \infty), \) and we get

\[ Im F_+^0 F_+ (I_{+}(t)S_0(t))(-u) = 2iLLS_0(t)(\cdot)(u), \]

\[ 2SiCoS_0(t)(\cdot)(u) = 2LLS_0(t)(\cdot)(u), u \in [0, \infty). \]

The existence of all the expressions of proof ensues from the 2 condition of theorem 1 ( with help of the formula of integration on parts with \( S_0(0) = 0 \) (the work of Fichtengoltz ([4]) ).

The theorem 1 is proved.

On the face of it such functions do not exist, but the class of functions of theorem 1 is not empty ( the lemma 2).

In the lemma 1 we use a class of functions \( G \).

By definition, \( S(t) \in G_1, \) if

1. \( S(p) \) is regular in all the complex plain with violation of analyticity at, perhaps, a finite number of points \( z_1, \ldots, z_k \notin (-\infty, \infty) \cup (-i\infty, i\infty); \)

2. \( \int_{-\infty}^\infty |S(t)| dt < \infty, S(0) = 0. \)

**Lemma 1.**

For \( S(p) \in G_1, S(-p) = -S(p), S(0) = 0, \)

\[ (1/2)LLS(\sqrt{t})(\cdot)(p^2) = (1/i)LF_+(S(t))(\cdot)(p), p \in (0, \infty), \]

and the \( LLS(\sqrt{t})(\cdot)(p) \) function is similar to the \( \sqrt{p} \) function.

**Proof.**

We obtain ( the work of Fichtengoltz ([4]) )

\[ (1/i)LF_+(S(t))(\cdot)(p) = \int_0^\infty S(t)[t/(p^2 + t^2)] dt = \]

\[ = (1/2) \int_0^\infty S(\sqrt{t})[1/((p^2 + t^2)] dt = \]

\[ LL(1/2)S(\sqrt{t})(\cdot)(p^2), p = x \in [0, \infty). \]

The behavior of the \( LLS(\sqrt{t})(\cdot)(p) \) function is a result of equality

\[ (1/i)LF_+(S(t))(\cdot)(-x) = \lim_{x+iw \rightarrow x, y>0} (LL)_{up}S(\sqrt{t})(\cdot)(p) = \]

\[ = \lim_{x+iw \rightarrow x, y<0} LL_{down}S(\sqrt{t})(\cdot)(p), \]

\( x \in (0, \infty), \) where we got the first limit as the analytic continuation of

\[ LLS(\sqrt{t})(\cdot)(p) = (LL)_{up}S(\sqrt{t})(\cdot)(p) \]

round 0 clockwise, and we got the second limit as the analytic continuation of

\[ LLS(\sqrt{t})(\cdot)(p) = LL_{down}S(\sqrt{t})(\cdot)(p) \]

round 0 anticlockwise ( the analytic continuation from the real axis \([0, \infty)\); we use, that the \( LF_\pm^0 (S(t))(\cdot)(p) \) function is regular in all the complex plain with violation of analyticity at, perhaps, a finite number of points \( z_1, \ldots, z_k \notin (-\infty, \infty) \cup (-i\infty, i\infty); \)
of analyticity at, perhaps, a finite number of points). The regularity of the $LF_\pm(S(t))(\cdot)(\cdot)$ function takes place, if $S(p) \in G_1$ (the remark 1).

**Remark 1.**

The $LF_\pm(S(t))(\cdot)(\cdot)$ function is regular in all the complex plain with violation of analyticity at, perhaps, a finite number of points $z_1, \ldots, z_k \notin (-\infty, \infty) \cup (-i\infty, i\infty)$, if $S(p) \in G_1$.

**Proof.**

We can write (the works of Flihtengoltz ([4]) and Pavlov ([2]))

$$LF_\pm S(t)(\cdot)(\cdot) = \int S(t)[1/(p+it)]dt, \quad p \in [0, \infty),$$

if $S(p) \in G_1$.

The proof of remark 1 we obtain from

$$LF_\pm S(t)(\cdot)(a+iu) - LF_\pm S(t)(\cdot)(-a+iu) = \frac{1}{a^2 + i^2} \int S(t_1 - u)|2a/(a^2 + i^2)|dt_1 =$$

$$= \int S(at - u)[2/(1 + t^2)]dt \rightarrow \pi S(-u), \quad a \rightarrow 0,$n

$u \in (-\infty, \infty)$, and

$$LF_\pm S(t)(\cdot)(p) - (LF_\pm)_{itf1} S(t)(\cdot)(p) = \pi S(-p), \quad p \in i(\infty, \infty),$$

where

$$(LF_\pm)_{itf1} S(t)(\cdot)(p) = LF_\pm S(t)(\cdot)(p), \quad Re p < 0,$$

and

$$\pi S(-p) + (LF_\pm)_{itf1} S(t)(\cdot)(p), \quad Re p < 0,$$

is the analytic continuation of the $LF_\pm S(t)(\cdot)(\cdot)$ function to the left-half plane, if $S(p) \in G_1$ (the work of Lavrentiev and Shabat ([5])).

The remark 1 is proved.

In the lemma 2 we use a $G_1(\pm)$ class of functions,

By definition, $S(t) \in G_1(\pm)$, if

1.

$$S(t) = \int e^{itx} S_1(x)dx,$$

the $dS_1^2(x)/dx^2$ function is continuous on $[-a, a]$;

2.

$$S_1(-a) = S_1(a) = 0.$$

**Lemma 2.**

The equality

$$f(p) = LF_\pm(S(t))(\cdot)(\cdot), \quad f(p) = f_1(-p), \quad p \in (0, \infty),$$

takes place, if $S(t) \in G_1(\pm), S(t) = S(t), \quad t \in [0, \infty),$, where the $f_1(p)$ function is the analytic continuation of the function $f(p), \quad Re p > 0$, to the left-half plane, and the function $f(p) = f_1(p)$ is regular in all the complex plane.

**Proof.**

From the definition of the $S(t)$ function and the inversion formula of Fourier transform (the work of Kolmogorov and Fomin ([6])) we obtain

$$f(p) = LF_\pm(S(t))(\cdot)(\cdot) =$$

$$= \int e^{-pu}du \int e^{ut} S(t)dt =$$

$$= 2\pi \int e^{-pu}du S_1(u)du,$$

as for $Re p > 0$, so as for all complex $p$ (from

$$df(p)/dp = 2\pi \int e^{-pu}S_1(u)du, \quad p \in C$$

the last integral is, obviously, regular in the complex plan).

We will prove the equality $f(-p) = f(p)$ without an application of the methods of function of integer type.

With help of $S_1(-a) = S_1(a) = 0$ we obtain

$$\int_{-\infty}^{\infty} |S(t)|dt \ll \infty,$$

( from the condition 2 of the $G_1(\pm)$ class (the work of Flihtengoltz ([4]) we get $|S(t)| < const./t^2, \quad t \rightarrow \pm\infty$ with help of the integration on parts).

After the change of limits of integration we get

$$f(p) = \int_{-\infty}^{\infty} e^{-(x+iy)t}du \int_{-\infty}^{\infty} e^{uit} S(t)dt =$$

$$= \int_{-\infty}^{\infty} S(t)[1/(p+it)]dt, \quad Re p = x > 0, \quad p = x + iy,$$

where

$$f(p) = \int_{-\infty}^{\infty} e^{-(x+iy)t}du \int_{-\infty}^{\infty} e^{uit} S(t)dt =$$

$$= \int_{-\infty}^{\infty} [-e^{-(x+iy)t}a + 1]S(t)[1/(x-iy+it)]dt, \quad Re p = x > 0.$$ We obtain

$$\lim_{x+iy \rightarrow x+iy} f(x+iy) = -\int_{-\infty}^{\infty} [-e^{-(x+iy+t)a} + 1]S(t)[1/(x+iy+it)]dt, \quad Re p = x > 0.$$ By analogy for $x < 0$, with help of regularity of the function $f_1(p), \quad Re p > 0, \quad Re p < 0$ in the first form with obvious $df(p)/dp, \quad Re p \neq 0$, and with help

$$\lim_{p\rightarrow a^+} f_1(p) = \lim_{p\rightarrow a^+} f_1(p), \quad a \in (-\infty, \infty),$$
(we use the work of Lavrentiev and Shabat ([5,p.147,])
), we obtain

\[
f_1(p) = -e^{-xa} \int_{-\infty}^{\infty} [e^{-iy+it}a + 1]S(t)[1/(-x+iy+it)]dt,
\]

\[ p = x + iy, \text{Re} \ p = x < 0, \text{where} \]

\[
f_1(p) = - \int_{-\infty}^{\infty} [e^{-(-x+iy)it}a + 1]S(t)[1/(-x+iy+it)]dt,
\]

\[ x \in (-\infty, 0), \]

where

\[
0 \equiv \int_{-\infty}^{\infty} [e^{-(-x+iy)it}aS(t)[1/(-x+iy+it)]dt = 0,
\]

\[ -x \in (-\infty, 0), \]

We obtain

\[
f(x) = - \int_{-\infty}^{\infty} S(t)[1/(-x+iy+it)]dt = f_1(-x),
\]

\[ -x \in (-\infty, 0), \text{if} \ S(-t) = -S(t), t \in [0, \infty). \]

The lemma 2 is proved.

By definition,

\[
I_0(t) = 1, t \in [-a, a], I_0(t) = 0, t \neq (-a, a), \]

\[
S_1(t) = S_1(t)I_0(t), t \in (-\infty, \infty), \]

\[
S_0(t) = \int_{-\infty}^{\infty} e^{int}S(u)du = \int_{-\infty}^{\infty} e^{iut}S(u)du,
\]

where from the inversion theorem for Fourier transform

( the works of Kolmogorov and Fomin ([6])
)

\[
S_0(t) = 2\pi S_1(t), t \in (-\infty, \infty). \]

Theorem 2.

1.

\[
\text{CoSiLLS}_0(t)(\cdot)(x) = (\pi/2)^2 S_0(t)(\cdot)(x), x \in (0, \infty), \]

if

\[
S_0(t) = \int_{-\infty}^{\infty} e^{iut}S(u)du, S(-u) = -S(u), u \in (0, \infty),
\]

\[ t \in (-\infty, \infty), S(p) \in G_1(*). \]

2.

\[
\text{CoSiLLS}_1(t)(\cdot)(x) = (\pi/2)^2 S_1(t)(\cdot)(x), x \in (0, \infty), \]

if \( S_1(t) \) and \( d^2S_1(t)/dt^2 \) are continuous on \([-a, a] \) ( with the points \(-a, a \) from the side of the interval \((-a, a) \)
), and, if

\[
S_1(a) = S(a) = S_1(0) = 0, S_1(-t) = -S_1(t), t \in [-a, a],
\]

( the \( S_1(t) \) function is considered in the definition of
class \( G_1(*) \), \( S_1(t) = S_1(t)I_0(t) = 2\pi S_0(t), t \in (-\infty, \infty) \)
).

Proof.

From the lemma 2 ( with \( f(p) = R(p) \) ) we obtain the condition 1 of theorem 1:

\[
R(is) = LF_+ S(x)(\cdot)(is) = \]

\[
LF_+ S(x)(\cdot)(-is) = R(-is), s \in (0, \infty).
\]

The \( R(p) \) function ( in lemma 2 \( R(p) = f(p) \) ) is regular in \( C \) ( the lemma 2 too), and we obtain the first main condition of theorem 1.

From the second condition of the class \( G_1(*) \) we obtain the condition 2 in theorem 2; we use the integration
on parts and \( S_0(0) = 0, S_0(-a) = S_0(a) = 0, S_0(x) = 0, |x| > a \), where

\[
S_0(t) = \int_{-\infty}^{\infty} e^{iut}S(u)du = \int_{-\infty}^{\infty} e^{iut}du \int_{-a}^{a} e^{iuv}S_1(v)dv =
\]

\[
2\pi S_1(-v) = -2\pi S_1(v), v \in (-\infty, \infty),
\]

\[ S_1(u) = 0, |u| > a > 0, \text{the work of Fihtengoltz([4])}. \]

From

\[
\int_{-\infty}^{\infty} |\text{CoSiLL}(t)(\cdot)(x)|dx < \infty,
\]

and

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S_0(x)|dx < \infty,
\]

with help of the inversion theorem for Fourier transform

( the works of Kolmogorov and Fomin ([6])
and Pavlov([7]) ) we get the first formula of theorem 2.

The second equality of theorem 2 we obtain from the inversion formula of Fourier transform :

\[
S_1(t)I_0(t) = \int_{-\infty}^{\infty} e^{iut}du \int_{-\infty}^{\infty} S_1(x)e^{-iux}dx,
\]

\[ t \in (-\infty, \infty), S_1(-a) = S_1(a) = S_1(0). \]
3 Examples

Example 1
We can write
\[ \int_0^\infty [udu/(x^2 + u^2)] \int_0^\infty S_0(t)[dt/(u^2 + t^2)] = \]
\[ = \frac{\pi}{2} L \int_0^\infty S_0(t)(x) \, dt = \pi x \int_0^\infty \left[ \frac{1}{(x^2 + u^2)} \right] \, du = \]
\[ = \frac{\pi}{2} S_0(x), \quad x \in (0, \infty), \quad S_0(0) = 0. \]
The expression of inlying integral in the terms of logarithms is more difficult for analysis.

Example 2
From the lemma 2 we got the interesting equality:
\[ -e^{ia} \int_{-\infty}^{\infty} e^{-ita} S(t)[1/(p + it)] \, dt = 0, \]
p = x ∈ (−∞, ∞), if
\[ S(t) = \int_a^\infty e^{itx} S_1(x) \, dx, \]
for all the functions S_1(x)
\[ S_1(0) = S_1(−a) = S_1(a) = 0, \]
if d^2 S_1(x)/dx^2 is continuous on [−a, a], for a > 0.

For instance, for S_1(x) = e^{−x^2} sin x we obtain
\[ \int_{-\infty}^{\infty} e^{-itx} S_1(x) \, dx = 0, \]
p = u ∈ (−∞, ∞), if
\[ S(t) = \int_0^{\pi/k} \sin tx e^{−x^2} \, dx, \]
\[ k = 0, \pm 1, \pm 2, \ldots. \]

4 Conclusions
The formulas of part 2 of theorem 2 facilitate the calculations, if we search the function S_1(t):
\[ L(x) = \int_0^\infty e^{-xt} S_1(t) \, dt, \quad x \in [0, \infty), \]
and, if we use only the values L(x): x ∈ [0, +∞). But the formula of theorem 2 requires the further researches, first of all, for the functions S_1(−a) = ±S_1(a) ≠ 0, S_1(−x) = S_1(x), x ∈ (−∞, ∞).

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