Modeling of Stochastic Processes in $L_p(T)$ Using Orthogonal Polynomials

Oleksandr Mokliachuk

National Technical University of Ukraine "Kyiv Polytechnic Institute", Kiev, Ukraine

*Corresponding Author: omoklyachuk@gmail.com

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Abstract In this paper, models that approximate stochastic processes from the space $Sub_\varphi(\Omega)$ with given reliability and accuracy in $L_p(T)$ are considered for some specific functions $\varphi(t)$. For processes that are decomposed in series using orthonormal bases, such models are constructed in the case where elements of such decomposition cannot be found explicitly.

Keywords Models of stochastic processes, φ -sub-Gaussian processes, orthogonal polynomials

1 Introduction

In different applications of the theory of stochastic processes it is vital to construct the model of the studied process. One way to construct the model of a stochastic process is to represent that process as the infinite series with respect to orthonormal polynomial basis and select the sum of first N elements of this series to be the model.

Consider stochastic process of the second order $X = \{X(t), t \in T\}$, $EX(t) = 0 \forall t \in T$, and let $B(t, s) = EX(t)X(s)$ be the correlation function of this stochastic process X . Then the next statement holds true.

Theorem 1 [1] (On decomposition of the stochastic process using an orthonormal basis) Let $X(t)$, $t \in T$ be stochastic process of the second order, $EX(t) = 0 \forall t \in T$, let $B(t, s) = EX(t)X(s)$ be the correlation function of X , let $f(t, \lambda)$ be some function from $L_2(\Lambda, \mu)$ space, and let $\{g_k(\lambda), k \in Z\}$ be the orthonormal basis in $L_2(\Lambda, \mu)$ space. Then, correlation function $B(t, s)$ is represented in the form

$$B(t, s) = \int_{\Lambda} f(t, \lambda) \overline{f(s, \lambda)} d\mu(\lambda)$$

if and only if the process can be represented in the form

$$X(t) = \sum_{k=1}^{\infty} a_k(t) \xi_k, \quad (1)$$

where

$$a_k(t) = \int_{\Lambda} f(t, \lambda) \overline{g_k(\lambda)} d\mu(\lambda), \quad (2)$$

ξ_k are centered uncorrelated random variables that satisfy the conditions: $E\xi_k = 0$, $E\xi_k \xi_l = \delta_{kl}$, $E\xi_k^2 = 1$.

When constructing models of the processes, it is difficult or impossible to find $a_k(t)$ explicitly. In that case, we have to use approximations of these elements. Let us now introduce the model of such process.

Definition 1 Let stochastic process $X = \{X(t), t \in T\}$ allow decomposition (1). We will call stochastic process $X_N = \{X_N(t), t \in T\}$ model of the process X , if

$$X_N(t) = \sum_{k=1}^N \xi_k \hat{a}_k(t), \quad (3)$$

where $\hat{a}_k(t)$ are approximations of functions $a_k(t)$ in the form (2), ξ_k are centered uncorrelated random variables, $E\xi_k = 0$, $E\xi_k \xi_l = \delta_{kl}$, $E\xi_k^2 = 1$.

We will consider in details the case where ξ_k are independent φ -sub-Gaussian random variables.

Since $\hat{a}_k(t)$ are approximations of functions $a_k(t)$, they will introduce some error in the model of stochastic process. The next theorem deals with that case.

Theorem 2 [2]

Let $X \in Sub_\varphi(\Omega)$, $X = \{X(t), t \in [0, T]\}$ be a stochastic process,

$$\varphi(t) = \begin{cases} \frac{t^2}{\gamma}, & t < 1 \\ \frac{t^\gamma}{\gamma}, & t \geq 1 \end{cases},$$

where $\gamma > 2$. Let

$$c_N = \int_0^T \left(\sum_{k=1}^N \tau_\varphi^2(\xi_k) \delta_k^2(t) + \sum_{k=N+1}^{\infty} \tau_\varphi^2(\xi_k) a_k^2(t) \right)^{p/2} dt < \infty.$$

Model X_N approximates stochastic process X with given reliability $1 - \alpha$ and accuracy δ in the space $L_p(0, T)$, if

$$\begin{cases} c_N \leq \delta / (\beta \ln \frac{2}{\alpha})^{p/\beta} \\ c_N < \delta / p^{p(1-1/\gamma)} \end{cases},$$

and β satisfies the inequality $1/\beta + 1/\gamma = 1$.

A system of orthonormal functions can be considered as basis $\{g_k(\lambda)\}$. It is interesting to consider bases that consist of sets of orthogonal polynomials. The classical examples of such polynomial sets are Chebyshev polynomials, Legendre polynomials, Hermite polynomials, Laguerre polynomials, Jacobi polynomials.

For a system of polynomials $\{P_n(x)\}$ that are orthogonal with the weight function $h(x)$ on some interval (a, b) and for some fixed $x \in (a, b)$ the next series can be considered:

$$GF(x, \omega) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} \omega^n.$$

Under some minimal conditions, this series has positive convergence radius. In such a case, the function $GF(x, \omega)$ is called generating function of the polynomial set $\{P_n(x)\}$ [3].

If for some functional basis $\{g_k(\lambda)\}$ a generating function exists, namely, if for some ω

$$GF_g(x, \omega) = \sum_{k=0}^{\infty} g_k(\lambda) \omega^k,$$

then, under additional condition $\tau_\varphi(\xi_k) = \tau \omega^k$, the next equality holds true for the process $X(t)$:

$$\begin{aligned} \tau_\varphi(X(t)) &= \tau_\varphi \left(\sum_{k=0}^{\infty} \xi_k \int_a^b f(t, \lambda) g_k(\lambda) d\lambda \right) = \\ &= \tau_\varphi \left(\int_a^b f(t, \lambda) \left(\sum_{k=0}^{\infty} \xi_k g_k(\lambda) \right) d\lambda \right) \leq \\ &\leq \tau \int_a^b f(t, \lambda) \left(\sum_{k=0}^{\infty} \omega^k g_k(\lambda) \right) d\lambda = \\ &= \tau \int_a^b f(t, \lambda) GF_g(\lambda, \omega) d\lambda. \end{aligned}$$

2 Modeling of stochastic processes in $L_p(0, T)$ using the Hermite polynomials

Let $X = \{X(t), t \in [0, T]\} \in Sub_\varphi(\Omega)$ be stochastic process of the second order, $EX(t) = 0$. Let the correlation function of the process X , $B(t, s) = EX(t)X(s)$, be represented as

$$B(t, s) = \int_{-\infty}^{\infty} f(t, \lambda) f(s, \lambda) d\lambda,$$

where $f(t, \lambda)$, $t \in [0, T]$, $\lambda \in R$ is a family of functions from $L_2(R)$. Since Hermite functions [4] form an orthonormal basis, the stochastic process X , according to the theorem 1, can be represented as

$$X(t) = \sum_{k=0}^{\infty} \xi_k \int_{-\infty}^{\infty} f(t, \lambda) \hat{H}_k(\lambda) d\lambda,$$

where ξ_k are centered uncorrelated random variables, $E\xi_k = 0$, $E\xi_k \xi_l = \delta_{kl}$, $E\xi_k^2 = 1$; $\hat{H}_k(\lambda)$ are Hermite functions:

$$\hat{H}_k(\lambda) = \frac{H_k(\lambda)}{\sqrt{k!}} \frac{1}{\sqrt[4]{2\pi}} \exp\left\{-\frac{\lambda^2}{2}\right\}, \quad (4)$$

where $H_k(\lambda)$ are the Hermite polynomials:

$$H_k(\lambda) = (-1)^k e^{\lambda^2/2} \frac{d^k}{d\lambda^k} e^{-\lambda^2/2}.$$

Theorem 3 Let a stochastic process $X = \{X(t), t \in [0, T]\}$ belong to the space $Sub_\varphi(\Omega)$ with

$$\varphi(t) = \begin{cases} \frac{t^2}{\gamma}, & t < 1 \\ \frac{t^\gamma}{\gamma}, & t \geq 1 \end{cases}$$

for $\gamma > 2$, let process $X(t)$ can be represented in the form (1), and let the series $\{\hat{H}_k(t)\}$ of Hermite functions be the basis. Let

$$\begin{aligned} c_N &= \int_0^T \left(\int_{-\infty}^{\infty} Z_f^2(t, \lambda) d\lambda \sum_{k=N+1}^{\infty} \frac{\tau_\varphi^2(\xi_k)}{k^2 + 3k + 2} + \right. \\ &\quad \left. + \sum_{k=1}^N \tau_\varphi^2(\xi_k) \delta_k^2(t) \right)^{p/2} dt < \infty, \end{aligned}$$

$$Z_f(t, \lambda) = \left| \frac{\partial^2 f(t, \lambda)}{\partial \lambda^2} - \lambda \frac{\partial f(t, \lambda)}{\partial \lambda} + \frac{\lambda^2 - 2}{4} f(t, \lambda) \right|,$$

where function $f(t, s)$ is twice differentiable and bounden with respect to the variable s , $Z_f(\lambda)$ is integrable on R . The model $X_N(t)$, defined in (3), approximates the stochastic process $X(t)$ with given reliability $1 - \alpha$ and accuracy δ in $L_p(0, T)$ spaces, if

$$\begin{cases} c_N \leq \delta / (\beta \ln \frac{2}{\alpha})^{p/\beta} \\ c_N < \delta / p^{p(1-1/\gamma)} \end{cases},$$

where $1/\gamma + 1/\beta = 1$.

Proof. According to the theorem conditions,

$$\begin{aligned} a_k(t) &= \int_{-\infty}^{\infty} f(t, \lambda) \hat{H}_k(\lambda) d\lambda = \\ &= \frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} f(t, \lambda) \frac{H_k(\lambda) e^{-\lambda^2/4}}{\sqrt{k!}} d\lambda. \end{aligned}$$

Using properties of Hermite polynomials[4], we can show that

$$\frac{\partial H_k(t)}{\partial t} = k H_{k-1}(t).$$

Using integration by parts, we get:

$$\begin{aligned} a_k(t) &= \frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} f(t, \lambda) \frac{H_k(\lambda) e^{-\lambda^2/4}}{\sqrt{k!}} d\lambda = \\ &= \frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} f(t, \lambda) \frac{e^{-\lambda^2/4}}{\sqrt{k+1} \sqrt{(k+1)!}} \frac{\partial H_{k+1}(\lambda)}{\partial \lambda} d\lambda = \\ &= \frac{1}{\sqrt[4]{\pi}} f(t, \lambda) \frac{e^{-\lambda^2/4}}{\sqrt{k+1} \sqrt{(k+1)!}} H_{k+1}(\lambda) \Big|_{\lambda=-\infty}^{\lambda=\infty} - \\ &\quad - \frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} \frac{\partial(f(t, \lambda) e^{-\lambda^2/4})}{\partial \lambda} \frac{H_{k+1}(\lambda)}{\sqrt{k+1} \sqrt{(k+1)!}} d\lambda. \end{aligned}$$

Since $H_k(\lambda) \exp\{-\lambda^2/4\}$ tends to zero as $\lambda \rightarrow \pm\infty$, and $f(t, \lambda)$ is bounded, we get

$$a_k(t) = -\frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} \frac{\partial(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda} \frac{H_{k+1}(\lambda)}{\sqrt{k+1}\sqrt{(k+1)!}} d\lambda.$$

Integration by parts one more time gives

$$\begin{aligned} a_k(t) &= -\frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} \frac{\partial(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda} \frac{H_{k+1}(\lambda)}{\sqrt{k+1}\sqrt{(k+1)!}} d\lambda = \\ &= -\frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} \frac{\partial(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda} \times \\ &\times \frac{1}{\sqrt{(k+1)(k+2)}\sqrt{(k+2)!}} \frac{\partial H_{k+2}(\lambda)}{\partial\lambda} d\lambda = \\ &= -\frac{1}{\sqrt[4]{\pi}} \frac{\partial(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda} \times \\ &\times \frac{1}{\sqrt{(k+1)(k+2)}\sqrt{(k+2)!}} H_{k+1}(\lambda) \Bigg|_{\lambda=-\infty}^{\lambda=\infty} + \\ &+ \frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} \frac{\partial^2(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda^2} \times \\ &\times \frac{H_{k+2}(\lambda)}{\sqrt{(k+1)(k+2)}\sqrt{(k+2)!}} d\lambda. \end{aligned}$$

It is certain that $H_{k+1}(\lambda)\partial(f(t, \lambda)e^{-\lambda^2/4})/\partial\lambda$ tends to zero as $\lambda \rightarrow \pm\infty$, since $\partial(f(t, \lambda)e^{-\lambda^2/4})/\partial\lambda = e^{-\lambda^2/4}(\partial f(t, \lambda)/\partial\lambda - \lambda f(t, \lambda))$, $H_{k+1}(\lambda)e^{-\lambda^2/4} \rightarrow 0$, and $\partial f(t, \lambda)/\partial\lambda - \lambda f(t, \lambda)$ is bounded, because $f(t, \lambda)$ and $\lambda f(t, \lambda)$ are bounded due to the conditions of the theorem. Then,

$$\begin{aligned} a_k(t) &= \frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} \frac{\partial^2(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda^2} \times \\ &\times \frac{H_{k+2}(\lambda)}{\sqrt{(k+1)(k+2)}\sqrt{(k+2)!}} d\lambda. \end{aligned}$$

Because $\hat{H}_k$ is an orthonormal basis, $\int_{-\infty}^{\infty} \hat{H}_k^2(t) dt = 1$. That's why

$$\begin{aligned} &\frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} \frac{\partial^2(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda^2} \times \\ &\times \frac{H_{k+2}(\lambda)}{\sqrt{(k+1)(k+2)}\sqrt{(k+2)!}} d\lambda = \\ &= \frac{1}{\sqrt{(k+1)(k+2)}} \int_{-\infty}^{\infty} \frac{\partial^2(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda^2} \times \\ &\times \frac{H_{k+2}(\lambda)e^{-\lambda^2/4}}{\sqrt{(k+2)!}\sqrt[4]{2\pi}} e^{\lambda^2/4} d\lambda = \\ &= \frac{1}{\sqrt{(k+1)(k+2)}} \int_{-\infty}^{\infty} \frac{\partial^2(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda^2} \times \\ &\times g_{k+2}(\lambda)e^{\lambda^2/4} d\lambda \leq \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\sqrt{(k+1)(k+2)}} \left(\int_{-\infty}^{\infty} \left(\frac{\partial^2(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda^2} \times \right. \right. \\ &\left. \left. \times e^{\lambda^2/4} \right)^2 d\lambda \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} g_{k+2}^2(\lambda) d\lambda \right)^{\frac{1}{2}} = \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sqrt{(k+1)(k+2)}} \left(\int_{-\infty}^{\infty} \left(\frac{\partial^2(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda^2} \times \right. \right. \\ &\left. \left. \times e^{\lambda^2/4} \right)^2 d\lambda \right)^{\frac{1}{2}} \end{aligned}$$

Besides,

$$\begin{aligned} &\frac{\partial^2(f(t, \lambda)e^{-\lambda^2/4})}{\partial\lambda^2} e^{\lambda^2/4} = \frac{\partial}{\partial\lambda} \left(\frac{\partial f(t, \lambda)}{\partial\lambda} e^{-\lambda^2/4} - \right. \\ &- \frac{1}{2} e^{-\lambda^2/4} \lambda f(t, \lambda) \Bigg) e^{\lambda^2/4} = \left(\frac{\partial}{\partial\lambda} \left(\frac{\partial f(t, \lambda)}{\partial\lambda} e^{-\lambda^2/4} - \right. \right. \\ &- \frac{\partial}{\partial\lambda} \left(\frac{1}{2} e^{-\lambda^2/4} \lambda f(t, \lambda) \right) \Bigg) e^{\lambda^2/4} = \left(\frac{\partial^2 f(t, \lambda)}{\partial\lambda^2} e^{-\lambda^2/4} - \right. \\ &- \frac{1}{2} e^{-\lambda^2/4} \lambda \frac{\partial f(t, \lambda)}{\partial\lambda} - \frac{1}{2} e^{-\lambda^2/4} \lambda \frac{\partial f(t, \lambda)}{\partial\lambda} - \\ &- f(t, \lambda) \frac{\partial}{\partial\lambda} \left(\frac{1}{2} e^{-\lambda^2/4} \lambda \right) \Bigg) e^{\lambda^2/4} = \left(\frac{\partial^2 f(t, \lambda)}{\partial\lambda^2} e^{-\lambda^2/4} - \right. \\ &- e^{-\lambda^2/4} \lambda \frac{\partial f(t, \lambda)}{\partial\lambda} - \\ &- f(t, \lambda) \left(\frac{1}{2} e^{-\lambda^2/4} - \frac{1}{4} e^{-\lambda^2/4} \lambda^2 \right) \Bigg) e^{\lambda^2/4} = \\ &= \frac{\partial^2 f(t, \lambda)}{\partial\lambda^2} - \lambda \frac{\partial f(t, \lambda)}{\partial\lambda} + \frac{\lambda^2 - 2}{4} f(t, \lambda). \end{aligned}$$

Finally, we obtain

$$a_k(t) \leq \left(\frac{1}{(k+1)(k+2)} \int_{-\infty}^{\infty} Z_f^2(t, \lambda) d\lambda \right)^{\frac{1}{2}},$$

where

$$Z_f(t, \lambda) = \left| \frac{\partial^2 f(t, \lambda)}{\partial\lambda^2} - \lambda \frac{\partial f(t, \lambda)}{\partial\lambda} + \frac{\lambda^2 - 2}{4} f(t, \lambda) \right|.$$

Therefore,

$$\begin{aligned} c_N &= \int_0^T \left(\sum_{k=1}^N \tau_{\varphi}^2(\xi_k) \delta_k^2(t) + \right. \\ &+ \left. \sum_{k=N+1}^{\infty} \tau_{\varphi}^2(\xi_k) a_k^2(t) \right)^{p/2} dt \leq \end{aligned}$$

$$\leq \int_0^T \left(\int_{-\infty}^{\infty} Z_f^2(t, \lambda) d\lambda \sum_{k=N+1}^{\infty} \frac{\tau_\varphi^2(\xi_k)}{k^2 + 3k + 2} + \sum_{k=1}^N \tau_\varphi^2(\xi_k) \delta_k^2(t) \right)^{p/2} dt.$$

Finally, the statement of this theorem is derived from the theorem 2.

Theorem 4 Let stochastic process $X = \{X(t), t \in [0, T]\}$ belong to the space $\text{Sub}_\varphi(\Omega)$,

$$\varphi(t) = \begin{cases} \frac{t^2}{\gamma}, & t < 1 \\ \frac{t^\gamma}{\gamma}, & t \geq 1 \end{cases}$$

for $\gamma > 2$, let the process $X(t)$ allow representation in the form (1), and the series $\hat{H}_k(t)$ of Hermite functions is the basis. Let $\tau_\varphi(\xi_k) = \tau\omega^k$, $|\omega| < 1$, and

$$c_N = \int_0^T \left(\frac{\tau}{\sqrt{(1-\omega^2)}} \left(\int_{-\infty}^{\infty} f^2(t, \lambda) d\lambda \right)^{1/2} - \sum_{k=0}^N \tau\omega^k \hat{a}_k(t) \right)^p dt < \infty.$$

Model $X_N(t)$, provided in (3), approximates $X(t)$ with given reliability $1 - \alpha$ and accuracy δ in the space $L_p(0, T)$, if

$$\begin{cases} c_N \leq \delta / (\beta \ln \frac{2}{\alpha})^{p/\beta} \\ c_N < \delta / p^{p(1-1/\gamma)} \end{cases},$$

where $1/\gamma + 1/\beta = 1$.

Proof. According to the statement of the theorem,

$$\begin{aligned} a_k(t) &= \int_{-\infty}^{\infty} f(t, \lambda) \hat{H}_k(\lambda) d\lambda = \\ &= \frac{1}{\sqrt[4]{\pi}} \int_{-\infty}^{\infty} f(t, \lambda) \frac{H_k(\lambda) e^{-\lambda^2/2}}{\sqrt{k!}} d\lambda. \end{aligned}$$

Using the properties of Hermite polynomials $\{H_k(\lambda)\}$, we get

$$GF_{H^2}(\lambda, \omega) = \sum_{k=1}^{\infty} \frac{H_k^2(\lambda)}{k! 2^k} \omega^k = \frac{1}{\sqrt{1-\omega^2}} \exp \left\{ \frac{2\lambda^2\omega}{1+\omega} \right\}.$$

Moreover, such series converges for $|\omega| < 1$. Under the conditions of the theorem, $\tau_\varphi(\xi_k) = \tau\omega^k$. That's why, for the process $X(t)$ the next condition holds true:

$$\begin{aligned} \tau_\varphi(X(t)) &= \tau_\varphi \left(\sum_{k=1}^{\infty} \xi_k \int_{-\infty}^{\infty} f(t, \lambda) \hat{H}_k(\lambda) d\lambda \right) \leq \\ &\leq \tau \int_{-\infty}^{\infty} f(t, \lambda) \sum_{k=0}^{\infty} \omega^k \hat{H}_k(\lambda) d\lambda \leq \\ &\leq \tau \left(\int_{-\infty}^{\infty} f^2(t, \lambda) d\lambda \right)^{1/2} \times \end{aligned}$$

$$\begin{aligned} &\times \left(\int_{-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{H_k(\lambda) \exp\{-\lambda^2/2\}}{\sqrt{k! 2^k \sqrt{\pi}}} \omega^k \right)^2 d\lambda \right)^{1/2} \leq \\ &\leq \tau \left(\int_{-\infty}^{\infty} f^2(t, \lambda) d\lambda \right)^{1/2} \times \\ &\times \left(\int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{H_k^2(\lambda) \exp\{-\lambda^2\}}{k! 2^k \sqrt{\pi}} \omega^{2k} d\lambda \right)^{1/2} = \\ &= \tau \left(\int_{-\infty}^{\infty} f^2(t, \lambda) d\lambda \right)^{1/2} \times \\ &\times \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} GF_{H^2}(\lambda, \omega^2) \exp\{-\lambda^2\} d\lambda \right)^{1/2} = \\ &= \tau \left(\int_{-\infty}^{\infty} f^2(t, \lambda) d\lambda \right)^{1/2} \times \\ &\times \left(\frac{1}{\sqrt{\pi(1-\omega^4)}} \int_{-\infty}^{\infty} \exp \left\{ \frac{2\lambda^2\omega}{1+\omega} \right\} \exp\{-\lambda^2\} d\lambda \right)^{1/2} = \\ &= \tau \left(\int_{-\infty}^{\infty} f^2(t, \lambda) d\lambda \right)^{1/2} \times \\ &\times \left(\frac{1}{\sqrt{\pi(1-\omega^4)}} \int_{-\infty}^{\infty} \exp \left\{ \lambda^2 \frac{\omega^2-1}{\omega^2+1} \right\} d\lambda \right)^{1/2} = \\ &= \tau \left(\int_{-\infty}^{\infty} f^2(t, \lambda) d\lambda \right)^{1/2} \times \\ &\times \left(\frac{1}{\sqrt{\pi(1-\omega^4)}} \sqrt{\pi \frac{1+\omega^2}{1-\omega^2}} \right)^{1/2} = \\ &= \frac{\tau}{\sqrt{(1-\omega^2)}} \left(\int_{-\infty}^{\infty} f^2(t, \lambda) d\lambda \right)^{1/2}. \end{aligned}$$

Given these considerations, the estimator of the model of the process will take the next form:

$$\begin{aligned} \tau_\varphi(X(t) - X_N(t)) &= \tau_\varphi \left(\sum_{k=0}^{\infty} \xi_k a_k(t) - \sum_{k=0}^N \xi_k \hat{a}_k(t) \right) = \\ &= \tau_\varphi \left(\sum_{k=0}^N \xi_k \delta_k(t) + \sum_{k=N+1}^{\infty} \xi_k a_k(t) \right) \leq \\ &\leq \sum_{k=0}^N \tau_\varphi(\xi_k) \delta_k(t) + \sum_{k=N+1}^{\infty} \tau_\varphi(\xi_k) a_k(t) = \\ &= \sum_{k=0}^{\infty} \tau\omega^k a_k(t) - \sum_{k=0}^N \tau\omega^k \hat{a}_k(t) \leq \\ &\leq \frac{\tau}{\sqrt{(1-\omega^2)}} \left(\int_{-\infty}^{\infty} f^2(t, \lambda) d\lambda \right)^{1/2} - \sum_{k=0}^N \tau\omega^k \hat{a}_k(t). \end{aligned}$$

Statement of this theorem follows from the theorem 2 and the last inequality.

3 Modeling of stochastic processes in $L_p(0, T)$ using the Chebyshev polynomials

Let the process $X(t)$ has the same properties as the process from the previous section. Let orthonormal Chebyshev polynomials be used as the basis:

$$\hat{T}_n(\lambda) = \sqrt{\frac{2}{\pi}} T_n(\lambda),$$

where

$$T_n(\lambda) = \cos(n \arccos \lambda).$$

In such a case we can proof the next theorem.

Theorem 5 Let a stochastic process $X = \{X(t), t \in [0, T]\}$ belong to the space $Sub_\varphi(\Omega)$ with

$$\varphi(t) = \begin{cases} \frac{t^2}{\gamma}, & t < 1 \\ \frac{t^\gamma}{\gamma}, & t \geq 1 \end{cases}$$

for $\gamma > 2$, let $X(t)$ is represented in the form (1), and let the system of orthonormal Chebyshev polynomials $\{\hat{T}_k(t)\}$ be used as the basis. Let $\tau_\varphi(\xi_k) = \tau\omega^k$, $0 < \omega < 1$, and

$$c_N = \int_0^T \left(\sqrt{\frac{2}{\pi}} \tau \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2} \sqrt{D_T(\omega)} - \sum_{k=0}^N \tau\omega^k \hat{a}_k(t) \right)^p dt < \infty,$$

$$D_T(\omega) = 2 \frac{1}{\omega(4 + 3\omega^2 + \omega^4)} (\omega(5 + 5\omega^2 + 2\omega^2) +$$

$$+ (4 + 7\omega^2 + 4\omega^4 + \omega^6) \ln\{(\omega^2 - \omega + 2)/(\omega^2 + \omega + 2)\}).$$

Model $X_N(t)$, determined in (3), approximates the process $X(t)$ with given reliability $1 - \alpha$ and accuracy δ in the space $L_p(0, T)$, if

$$\begin{cases} c_N \leq \delta / (\beta \ln \frac{2}{\alpha})^{p/\beta} \\ c_N < \delta / p^{p(1-1/\gamma)} \end{cases},$$

where $1/\gamma + 1/\beta = 1$.

Proof. Under the conditions of the theorem,

$$a_k(t) = \int_{-1}^1 f(t, \lambda) \hat{T}_n(\lambda) d\lambda = \int_{-1}^1 f(t, \lambda) \sqrt{\frac{2}{\pi}} T_n(\lambda) d\lambda.$$

The generating function of orthogonal Chebyshev polynomials of the first kind $\{T_k(\lambda)\}$ has the next form:

$$GF_T(\lambda, \omega) = \sum_{k=0}^{\infty} T_k(\lambda) \omega^k = \frac{1 - \omega\lambda}{2 - \omega\lambda + \omega^2}$$

for $0 < \omega < 1$. Under the conditions of the theorem, $\tau_\varphi(\xi_k) = \tau\omega^k$. That's why for the process $X(t)$ the next condition is true:

$$\begin{aligned} \tau_\varphi(X(t)) &= \tau_\varphi \left(\sum_{k=1}^{\infty} \xi_k \int_{-1}^1 f(t, \lambda) \hat{T}_k(\lambda) d\lambda \right) \leq \\ &\leq \tau \int_{-1}^1 f(t, \lambda) \sum_{k=0}^{\infty} \omega^k \hat{T}_k(\lambda) d\lambda \leq \\ &\leq \tau \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2} \times \\ &\times \left(\int_{-1}^1 \left(\sum_{k=0}^{\infty} \sqrt{\frac{2}{\pi}} T_k(\lambda) \right)^2 d\lambda \right)^{1/2} = \\ &= \sqrt{\frac{2}{\pi}} \tau \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2} \times \\ &\times \left(\int_{-1}^1 \left(\frac{1 - \omega\lambda}{2 - \omega\lambda + \omega^2} \right)^2 d\lambda \right)^{1/2}. \end{aligned}$$

Lets calculate the second integral of the last expression separately.

$$\begin{aligned} &\int_{-1}^1 \left(\frac{1 - \omega\lambda}{2 - \omega\lambda + \omega^2} \right)^2 d\lambda = \\ &= 2 \frac{1}{\omega(4 + 3\omega^2 + \omega^4)} (\omega(5 + 5\omega^2 + 2\omega^2) + \\ &+ (4 + 7\omega^2 + 4\omega^4 + \omega^6) \ln\{(\omega^2 - \omega + 2)/(\omega^2 + \omega + 2)\} + \\ &+ \omega + 2) := D_T(\omega). \end{aligned}$$

Then the estimator of $\tau_\varphi(X(t))$ will take the form:

$$\tau_\varphi(X(t)) \leq \sqrt{\frac{2}{\pi}} \tau \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2} \sqrt{D_T(\omega)}.$$

Given these considerations, the estimator of the model of the process will take the following form:

$$\begin{aligned} \tau_\varphi(X(t) - X_N(t)) &= \tau_\varphi \left(\sum_{k=0}^{\infty} \xi_k a_k(t) - \sum_{k=0}^N \xi_k \hat{a}_k(t) \right) = \\ &= \tau_\varphi \left(\sum_{k=0}^N \xi_k \delta_k(t) + \sum_{k=N+1}^{\infty} \xi_k a_k(t) \right) \leq \\ &\leq \sum_{k=0}^N \tau_\varphi(\xi_k) \delta_k(t) + \sum_{k=N+1}^{\infty} \tau_\varphi(\xi_k) a_k(t) = \\ &= \sum_{k=0}^{\infty} \tau\omega^k a_k(t) - \sum_{k=0}^N \tau\omega^k \hat{a}_k(t) \leq \\ &\leq \sqrt{\frac{2}{\pi}} \tau \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2} \sqrt{D_T(\omega)} - \sum_{k=0}^N \tau\omega^k \hat{a}_k(t). \end{aligned}$$

Statement of this theorem follows from the theorem 2 and the last inequality.

We can also proof the similar theorem in the case of Chebyshev polynomials of the second kind:

$$U_n(\lambda) = \frac{\sin((n+1)\arccos \lambda)}{\sqrt{1-\lambda^2}}$$

$$\hat{U}_n(\lambda) = \sqrt{\frac{2}{\pi}} U_n(\lambda).$$

Generating function of the series $\{U_n(\lambda)\}$ has the form

$$GF_U(\lambda, \omega) = \sum_{k=0}^{\infty} \omega^k U_k(\lambda).$$

Let the process $X(t)$ has the same properties as in the previous theorem. Then, the next proposition holds true.

Theorem 6 *Let stochastic process $X = \{X(t), t \in [0, T]\}$ belong to the space $Sub_\varphi(\Omega)$,*

$$\varphi(t) = \begin{cases} \frac{t^2}{\gamma}, & t < 1 \\ \frac{t^\gamma}{\gamma}, & t \geq 1 \end{cases}$$

for $\gamma > 2$, let process $X(t)$ can be represented in the form (1), and let series $\{\hat{U}_k(t)\}$ of orthonormal Chebyshev polynomials be the basis. Let also $\tau_\varphi(\xi_k) = \tau\omega^k$, $0 < \omega < 1$, and

$$c_N = \int_0^T \left(\frac{2\tau}{\sqrt{\pi}(\omega^2 - 1)} \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2} - \sum_{k=0}^N \tau\omega^k \hat{a}_k(t) \right)^p dt < \infty.$$

Model $X_N(t)$, determined in (3), approximates the process $X(t)$ with given reliability $1 - \alpha$ and accuracy δ in the space $L_p(0, T)$, if

$$\begin{cases} c_N \leq \delta / (\beta \ln \frac{2}{\alpha})^{p/\beta} \\ c_N < \delta / p^{p(1-1/\gamma)} \end{cases},$$

where $1/\gamma + 1/\beta = 1$.

Proof. Under the conditions of the theorem, we have:

$$a_k(t) = \int_{-1}^1 f(t, \lambda) \hat{U}_n(\lambda) d\lambda = \int_{-1}^1 f(t, \lambda) \sqrt{\frac{2}{\pi}} U_n(\lambda) d\lambda.$$

Generation function of the series of orthogonal Chebyshev polynomials of the second kind $\{U_k(\lambda)\}$ has the following form:

$$GF_T(\lambda, \omega) = \sum_{k=0}^{\infty} U_k(\lambda) \omega^k = \frac{1}{1 - 2\omega\lambda + \omega^2}$$

$0 < \omega < 1$. As stated in the theorem's conditions, $\tau_\varphi(\xi_k) = \tau\omega^k$. That's why for stochastic process $X(t)$ the next statement holds true:

$$\tau_\varphi(X(t)) = \tau_\varphi \left(\sum_{k=1}^{\infty} \xi_k \int_{-1}^1 f(t, \lambda) \hat{U}_k(\lambda) d\lambda \right) \leq$$

$$\begin{aligned} &\leq \tau \int_{-1}^1 f(t, \lambda) \sum_{k=0}^{\infty} \omega^k \hat{U}_k(\lambda) d\lambda \leq \\ &\leq \tau \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2} \times \\ &\times \left(\int_{-1}^1 \left(\sum_{k=0}^{\infty} \sqrt{\frac{2}{\pi}} U_k(\lambda) \right)^2 d\lambda \right)^{1/2} = \\ &= \sqrt{\frac{2}{\pi}} \tau \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2} \times \\ &\times \left(\int_{-1}^1 \left(\frac{1}{1 - 2\omega\lambda + \omega^2} \right)^2 d\lambda \right)^{1/2} = \\ &= \sqrt{\frac{2}{\pi}} \tau \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2} \frac{\sqrt{2}}{(\omega^2 - 1)} = \\ &= \frac{2\tau}{\sqrt{\pi}(\omega^2 - 1)} \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2}. \end{aligned}$$

Taking into account these considerations, the estimator of the model of the stochastic process will fit the next condition:

$$\begin{aligned} \tau_\varphi(X(t) - X_N(t)) &= \tau_\varphi \left(\sum_{k=0}^{\infty} \xi_k a_k(t) - \sum_{k=0}^N \xi_k \hat{a}_k(t) \right) = \\ &= \tau_\varphi \left(\sum_{k=0}^N \xi_k \delta_k(t) + \sum_{k=N+1}^{\infty} \xi_k a_k(t) \right) \leq \\ &\leq \sum_{k=0}^N \tau_\varphi(\xi_k) \delta_k(t) + \sum_{k=N+1}^{\infty} \tau_\varphi(\xi_k) a_k(t) = \\ &= \sum_{k=0}^{\infty} \tau\omega^k a_k(t) - \sum_{k=0}^N \tau\omega^k \hat{a}_k(t) \leq \\ &\leq \frac{2\tau}{\sqrt{\pi}(\omega^2 - 1)} \left(\int_{-1}^1 f^2(t, \lambda) d\lambda \right)^{1/2} - \sum_{k=0}^N \tau\omega^k \hat{a}_k(t). \end{aligned}$$

Statement of this theorem follows from the theorem 2 and the last inequality.

4 Conclusions

Theorems are proved that allow to construct models of stochastic processes from $Sub_\varphi(\Omega)$ in the case where these processes are represented in series with respect to some orthogonal polynomial systems. The case when functional components of such decompositions cannot be found explicitly is studied.

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