Partial Derivatives of Three Variables Functions

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Abstract
This paper takes the mathematical software Maple as the auxiliary tool to study the partial differential problem of two types of three variables functions. We can obtain the infinite series forms of any order partial derivatives of these two types of functions by using differentiation term by term theorems, and hence greatly reduce the difficulty of calculating higher order partial derivative values of these functions. On the other hand, we propose two examples to do calculation practically.

Keywords Partial Derivatives, Infinite Series Forms, Differentiation Term By Term Theorem, Maple

1. Introduction
In calculus and engineering mathematics curricula, evaluating the \( n \)-th order partial derivative value of a multivariable function at some point, in general, needs to go through two procedures: firstly determining the \( n \)-th order partial derivative of this function, and then taking the point into this \( n \)-th order partial derivative. These two procedures will make us face with increasingly complex calculations when calculating higher order partial derivative values (i.e. \( n \) is large), and hence to obtain the answers by manual calculations is not easy. In this paper, we study the partial differential problem of the following two types of three variables functions

\[
\begin{align*}
 f(x, y, t) &= \ln[e^{2ax} - 2e^{ax}y^b \cos(ct + d) + y^{2b}] \\
 g(x, y, t) &= \tan^{-1}\left[\frac{y^b \sin(ct + d)}{e^{ax} - y^b \cos(ct + d)}\right]
\end{align*}
\]

Where \( a, b, c, d \) are real numbers. We can obtain the infinite series forms of any order partial derivatives of these two types of functions by using differentiation term by term theorem; these are the major results of this study (i.e., Theorems 1 and 2), and hence greatly reduce the difficulty of calculating their higher order partial derivative values. For the study of related partial differential problems can refer to [1-12]. In addition, we provide two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps to modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results
Firstly, we introduce a definition and some notations, formulas used in this paper.

2.1. Definition
The complex natural logarithm function \( \ln z \) is defined by \( \ln z = \ln|z| + i\theta \), where \( z \) is a complex number, \( \theta \) is a real number, and \( z = |z|e^{i\theta}, -\pi \leq \theta < \pi \).

2.2. Notations
2.2.1. Suppose \( r \) is any real number, \( m \) is any positive integer. Define \( \binom{m}{r} = \frac{m!}{r!(m-r)!} \), and \( (r)_0! = 1 \).
2.2.2. Let \( z = a + ib \) be a complex number, where \( i = \sqrt{-1}, a, b \) are real numbers. We denote \( a \) the real part of \( z \) by \( \text{Re}(z) \), and \( b \) the imaginary part of \( z \) by \( \text{Im}(z) \).
2.2.3. Assume \( p, q, r \) are non-negative integers. For the three variables function \( f(x, y, t) \), the \( p \) times partial derivative with respect to \( x \), \( q \) times partial derivative with respect to \( y \), and \( r \) times partial derivative with respect to \( t \), forms a \( p + q + r \)-th order partial derivative of \( f(x, y, t) \), and denoted by

\[
\partial^{p+q+r} f(x, y, t)
\]

2.3. Formulas
2.3.1. Euler’s formula

\[
f(x, y, t) = \ln[e^{2ax} - 2e^{ax}y^b \cos(ct + d) + y^{2b}] \quad (1)
g(x, y, t) = \tan^{-1}\left[\frac{y^b \sin(ct + d)}{e^{ax} - y^b \cos(ct + d)}\right] \quad (2)
\]
\( e^{i\alpha} = \cos \alpha + i \sin \alpha \), where \( \alpha \) is any real number.

### 2.3.2. DeMoivre’s Formula

\((\cos \alpha + i \sin \alpha)^n = \cos n\alpha + i \sin n\alpha\), where \( n \) is any integer, and \( \alpha \) is any real number.

### 2.3.3. \( \ln(1 - z) = -\sum_{k=1}^{\infty} \frac{z^k}{k} \), where \( z \) is a complex number, \( |z| < 1 \).

Next, we introduce an important theorem used in this study.

### 2.4. Differentiation term by term theorem (113)

If, for all non-negative integer \( k \), the functions \( g_k: (a, b) \to R \) satisfy the following three conditions:

1. There exists a point \( x_0 \in (a, b) \) such that \( \sum_{k=0}^{\infty} g_k(x_0) \) is convergent,
2. All functions \( g_k(x) \) are differentiable on open interval \( (a, b) \), and
3. \( \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x) \) is uniformly convergent on \( (a, b) \). Then \( \sum_{k=0}^{\infty} g_k(x) \) is uniformly convergent and differentiable on \( (a, b) \). Moreover, its derivative \( \frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x) \).

Before deriving the major results in this study, we need a lemma.

### 2.5. Lemma A

Suppose \( \lambda, \alpha \) are real numbers, \( |\lambda| < 1 \).

Then

\[
\ln(1 - \lambda e^{i\alpha}) = \frac{1}{2} \ln(1 - 2\lambda \cos \alpha + \lambda^2) - i \tan^{-1}\left(\frac{\lambda \sin \alpha}{1 - \lambda \cos \alpha}\right)
\]

### 2.5.1. Proof

\[
\ln(1 - \lambda e^{i\alpha}) = \ln(1 - \lambda \cos \alpha - i\lambda \sin \alpha) \quad \text{(By Euler’s formula)}
\]

\[
= \ln\left[\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}\right] - i \frac{\lambda \sin \alpha}{\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}}
\]

\[
= \ln\left[\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}\right] + \left(\frac{1 - \lambda \cos \alpha}{\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}}\right) - i \frac{\lambda \sin \alpha}{\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}}
\]

\[
= \frac{1 - \lambda \cos \alpha}{\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}} + i \frac{\lambda \sin \alpha}{\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}}
\]

\[
= \ln\left[\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}\right] + \left(\frac{1 - \lambda \cos \alpha}{\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}}\right) - i \frac{\lambda \sin \alpha}{\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}}
\]

### 2.6. Theorem 1

Let \( \alpha, \beta, \gamma, \delta \) be real numbers, \( p, q, r \) be non-negative integers. Suppose the domain of the three variables function

\[
f(x, y, t) = \ln[e^{\alpha x} - 2e^{\alpha y} \cos(\beta t + d) + e^{\gamma y + \delta t}] = \ln\left[\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}\right] + \left(\frac{1 - \lambda \cos \alpha}{\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}}\right) - i \frac{\lambda \sin \alpha}{\sqrt{(1 - \lambda \cos \alpha)^2 + (\lambda \sin \alpha)^2}}
\]

is \( x, y, t \in \mathbb{R}^3 \) and \( y < 0, |y| < e^{\alpha x} \).

Then the \( p + q + r \)-th order partial derivative of \( f(x, y, t) \),

\[
\frac{\partial^{p+q+r} f}{\partial x^p \partial y^q \partial t^r}(x, y, t) = 2a \cdot (1) x^{1-p} - 2(-1)^p a \cdot c e^{\mathcal{R}} \sum_{k=1}^{\infty} k^{p+q+r-1} \left(bk_q e^{-akx} y^{bk-q} \cos\left[k(\beta t + d) + \frac{\mathcal{R} \pi}{2}\right]\right)
\]

### 2.6.1. Proof

Let \( \lambda, \alpha \) be real numbers, \( |\lambda| < 1 \).

Because

\[
\frac{1}{2} \ln(1 - 2\lambda \cos \alpha + \lambda^2)
\]

\[
= \text{Re}[\ln(1 - \lambda e^{i\alpha})] \quad \text{(By Lemma A)}
\]

\[
= \text{Re}\left[-\sum_{k=1}^{\infty} \lambda^k e^{i\alpha k}\right] \quad \text{(By Formula 2.3.3)}
\]

\[
= \text{Re}\left[-\sum_{k=1}^{\infty} \lambda^k \cos k\alpha\right] \quad \text{(By DeMoivre’s formula)}
\]

\[
= -\sum_{k=1}^{\infty} \lambda^k \cos k\alpha \quad \text{(By Euler’s formula)}
\]
Taking \( \lambda = \frac{y^b}{e^{ax}} \), \( \alpha = ct + d \) into (5), we obtain
\[
f(x,y,t) = \ln[ e^{2ax} - 2e^{ax}y^b \cos(ct + d) + y^{2h} ]
\]
\[
= 2ax - 2 \sum_{k=1}^{\infty} \frac{1}{k} e^{-akx} y^{bk} \cos(k(ct + d))
\] (6)

Thus, by differentiation term by term theorem, differentiating \( p \) times with respect to \( x \), \( q \) times with respect to \( y \), and \( r \) times with respect to \( t \) on both sides of (6), we obtain the \( p + q + r \)-th order partial derivative of \( f(x,y,t) \),
\[
\frac{\partial^{p+q+r} f}{\partial t^p \partial y^q \partial x^r}(x,y,t) = 2a \cdot (1-p) x^{1-p} - 2(-1)^p a^p c^r \times
\]
\[
\sum_{k=1}^{\infty} k^{p+r-1} (bk)_q e^{-akx} y^{bk-q} \cos[k(ct + d) + \frac{r\pi}{2}]
\] (7)

The following is the second major result in this paper, we obtain the infinite series forms of any order partial derivatives of the three variables function (2).

2.7. Theorem 2

If the assumptions are the same as Theorem 1. Suppose the domain of the three variables function
\[
g(x,y,t) = \tan^{-1}\left[ \frac{y^b \sin(ct + d)}{e^{ax} - y^b \cos(ct + d)} \right]
\]
is
\[
\{(x,y,t) \in \mathbb{R}^3 | y^b \text{ exists}, y \neq 0, |y^b| < e^{ax} \}.
\]
Then the \( p + q + r \)-th order partial derivative of \( g(x,y,t) \),
\[
\frac{\partial^{p+q+r} g}{\partial t^p \partial y^q \partial x^r}(x,y,t) = (-1)^p a^p c^r \times
\]
\[
\sum_{k=1}^{\infty} k^{p+r-1} (bk)_q e^{-akx} y^{bk-q} \sin[k(ct + d) + \frac{r\pi}{2}]
\] (8)

3. Examples

In the following, for the partial differential problem of the two types of three variables functions in this study, we provide two examples and use Theorems 1, 2 to determine the infinite series forms of any order partial derivatives and some higher order partial derivative values of these functions. On the other hand, we employ Maple to calculate the approximations of these higher order partial derivative values and their infinite series forms for verifying our answers.

3.1. Example 1

Suppose the domain of the three variables function
\[
f(x,y,t) = \ln\left[ e^{8x} - 2e^{4x}y^{7/3} \cos\left(5t + \frac{\pi}{4}\right) + y^{14/3} \right]
\] (10)
is
\[
\{(x,y,t) \in \mathbb{R}^3 | y \neq 0, |y^{7/3}| < e^{4x} \} \text{ (the case of}
\]
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\[ a = 4, b = \frac{7}{3}, c = 5, d = \frac{\pi}{4} \text{ in Theorem 1} \]

Using Theorem 1, we obtain any \( p + q + r \)-th order partial derivative of \( f(x, y, t) \),

\[
\frac{\partial^{p+q+r} f}{\partial t^p \partial y^q \partial x^r}(x, y, t) = 8 \cdot (1)^{p-1} - 2(-1)^p 4^p 5^r \times \sum_{k=1}^{\infty} k^{p+r-1} \left( \frac{7k}{3} \right) e^{-4kx} y^{7k/3} \cos \left[ k \left( \frac{5t + \pi}{4} \right) + \frac{r \pi}{2} \right]
\]

Therefore, we can determine the 11-th order partial derivative value of \( f(x, y, t) \) at \( \left( \frac{2}{3}, \frac{\pi}{4} \right) \)

\[
\frac{\partial^{11} f}{\partial t^2 \partial y^2 \partial x^4}\left( \frac{2}{3}, \frac{\pi}{4} \right) = -2 \cdot 4^4 \cdot 5^4 \cdot \sum_{k=1}^{\infty} k^7 \left( \frac{7k}{3} \right) e^{-8k} \cdot 3^{7k/3} \cdot 3 \cos \left( \frac{5k \pi}{2} \right)
\]

Next, we use Maple to verify the correctness of (12).

\[ f := (x, y, t) \rightarrow \ln(e^{\exp(8x)} - 2 \cdot \exp(4x) \cdot y^{9/4} \cdot \cos(3t + 5\pi/6)) \]

\[ g := (x, y, t) \rightarrow \arctan(\frac{y^{9/4} \cdot \sin(3t - 5\pi/6)}{\exp(-6x) - y^{9/4} \cdot \cos(3t - 5\pi/6)}) \]

We also use Maple to verify the correctness of (15).

> evalf(D[1$3,2$6,3$4](g)(-2, 7, -5*Pi/6), 18);

\[ g := (x, y, t) \rightarrow \arctan(\frac{y^{9/4} \cdot \sin(3t - 5\pi/6)}{\exp(-6x) - y^{9/4} \cdot \cos(3t - 5\pi/6)}) \]

4. Conclusion

As mentioned, the differentiation term by term theorem plays a significant role in the theoretical inferences of this study. In fact, the application of this theorem is extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications.

On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

REFERENCES


159-173, 1992.


