Cone Metric Version of Existence and Convergence for Best Proximity Points

Byung-Soo Lee

Kyungsung University, Busan 608-736, Korea

*Corresponding Author: bslee@ks.ac.kr

Copyright ©2014 Horizon Research Publishing All rights reserved.

Abstract In 2011, Gabeleh and Akhar [3] introduced semi-cyclic-contraction and considered the existence and convergence results of best proximity points in Banach spaces. In this paper, the author introduces a cone semi-cyclic φ -contraction pair in cone metric spaces and considers best proximity points for the pair in cone metric spaces. His results generalize the corresponding results in [1–5].

Keywords a cone semi-cyclic φ -contraction pair, best proximity point, cone metric space, Banach space

2010 Mathematic Subject Classification: 47H10, 54H25

1. Introduction and Preliminaries

The existence and the convergence of best proximity points may be very applicable in nonlinear analysis including optimization problems by considering the strong applications of fixed point theory.

A cone metric space is a generalization of a metric space by replacing the real numbers with Banach spaces ordered by the given cone [4,6]. Generally, it is not possible to find the exact solution to the equation $Tx = x$ for a non-self mapping $T : A \rightarrow B$ defined on a cone metric space (X, d) , where A and B are nonempty subsets of (X, d) . Hence to focus the study on the problem of finding an element x which is the closest proximity to Tx is considerable in optimization senses. To consider a global minimization problem for a mapping $G : A \rightarrow (E, P)$, a Banach space ordered by a given cone P , defined by $G(x) = d(x, Tx)$ due to the fact that $d(A, B) \leq d(x, Tx)$ for all $x \in A$, is very reasonable and interesting.

In 2007, Al-Thagafi and Shahzad [1] introduced a class of cyclic φ -contractions in metric spaces which contains a class of cyclic contractions as a subclass, introduced by Eldred and Veeramani [2]. They obtained convergence and existence results of best proximity points for a cyclic φ -contractions in metric spaces and proved the existence of a best proximity point for a cyclic contraction in a reflexive Banach spaces which provide a positive answer to Eldred and Veeramani's question [2].

In 2011, Gabeleh and Akhar [3] introduced semi-cyclic-contraction and considered the existence and convergence results of best proximity points in Banach spaces.

In this paper, we introduce a cone semi-cyclic φ -contraction pair and obtain the existence and convergence results for the pair in cone metric spaces. Our results generalize the corresponding results in [1–5].

A nonempty subset P of a real Banach space E is called a (pointed) cone if and only if (P1) P is closed, $P \neq \{0\}$, (P2) for $a, b \in \mathbb{R}$ with $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$ and (P3) $x \in P$ and $-x \in P$ implies $x = 0$.

We define a partial ordering ' $\preceq$ ' with respect to P as follows; for $x, y \in E$, we say that $x \preceq y$ if and only if $y - x \in P$, $x \ll y$ if and only if $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P , $x \prec y$ if and only if $x \preceq y$ and $x \neq y$. A cone P is said to be normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

Let M be a nonempty set and (E, P) a Banach space with a given cone P . A mapping $d : M \times M \rightarrow (E, P)$ satisfying the conditions (d1) $0 \preceq d(x, y)$ for all $x, y \in M$ and $d(x, y) = 0$ if and only if $x = y$, (d2) $d(x, y) = d(y, x)$ for all $x, y \in M$ and (d3) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in M$ is called a cone metric on M and (M, d) is called a cone metric space.

Definition 1.1. [5]. Let (M, d) be a cone metric space. A subset A of M is said to be bounded above if there exists $c \in \text{int}P$ such that $c - d(x, y) \in P$ for all $x, y \in A$, and is said to be bounded if $\delta(A) = \sup\{d(x, y) : x, y \in A\}$ exists in E .

Let $\{x_n\}$ be a sequence in (M, d) and $x \in (M, d)$. If for every $c \in \text{int}P$, there is a natural number N such that for all $n > N$, $c - d(x_n, x) \in \text{int}P$, then we say that $\{x_n\}$ converges to x with respect to P and denote as $\lim_{n \rightarrow \infty} x_n = x$.

Lemma 1.1. [4] Let P be a normal cone. Let $\{x_n\}$ and $\{y_n\}$ be sequences in (M, d) .

- (i) $\{x_n\}$ converges to x with respect to P if and only if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) If $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$ with respect to P , then $d(x_n, y_n) \rightarrow d(x, y)$ as $n \rightarrow \infty$.
- (iii) If $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$ with respect to P and $y_n - x_n \in P$ for all $n \in \mathbb{N}$, then $y - x \in P$.

2. Properties of Constructed Sequences

Let A and B be nonempty subsets of a cone metric space (M, d) and $S, T : A \cup B \rightarrow A \cup B$ be mappings defined by $S(A \cup B) \subseteq B$ and $T(A \cup B) \subseteq A$.

Definition 2.1. A pair (S, T) is called a cone semi-cyclic φ -contraction pair if for a strictly increasing mapping $\varphi : E \rightarrow E$,

$$d(Sx, Ty) \preceq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)) \text{ for } x \in A \text{ and } y \in B.$$

When $S = T$, T is called a cone cyclic φ -contraction, where $d(A, B) = \inf_{\substack{x \in A \\ y \in B}} d(x, y)$.

Example 2.1. Put $(M, d) = (\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}, |\cdot|) \subset \mathbb{R}^2$, $E = \mathbb{R}$ and $P = (-\infty, 0]$. Let $A = \{(x_1, x_2) \mid 0 \leq x_1 \leq 4, x_2 \geq 4\}$, $B = \{(x_1, x_2) \mid x_1 \geq 4, 0 \leq x_2 \leq 4\}$ be subsets of (M, d) and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $\varphi(t) = t$ for $t \in \mathbb{R}$.

Define mappings $S, T : A \cup B \rightarrow A \cup B$ by

$$S(x_1, x_2) = \begin{cases} (x_2, x_1) & \text{for } (x_1, x_2) \in A \\ (x_1, x_2) & \text{for } (x_1, x_2) \in B, \end{cases}$$

$$T(x_1, x_2) = \begin{cases} (x_1, x_2) & \text{for } (x_1, x_2) \in A \\ (x_2, x_1) & \text{for } (x_1, x_2) \in B, \end{cases}$$

then $S(A \cup B) \subset B$ and $T(A \cup B) \subset A$.

Moreover,

$$\begin{aligned} & d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)) - d(Sx, Ty) \\ &= d(x, y) - d(x, y) + 0 - d(Sx, Ty) \\ &= -d(Sx, Ty) \\ &\leq 0 \text{ for } x \in A \text{ and } y \in B. \end{aligned}$$

Hence

$$d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)) - d(Sx, Ty) \in P,$$

which implies that

$$d(Sx, Ty) \preceq d(x, y) + \varphi(d(x, y)) + \varphi(d(A, B)).$$

Thus (S, T) is a cone semi-cyclic φ -contraction pair.

Remark 2.1. (i) A cone semi-cyclic k -contraction pair is a cone semi-cyclic φ -contraction pair with $\varphi(x) = (1 - k)x$ for $x \in E$. In this case the pair (S, T) satisfies for some $k \in (0, 1)$

$$d(Sx, Ty) \preceq k \cdot d(x, y) + (1 - k) \cdot d(A, B) \text{ for all } x \in A \text{ and } y \in B.$$

(ii) When $S = T$, T is called a cone cyclic k -contraction.

Sequences Construction For a cone semi-cyclic φ -contraction pair (S, T) , letting $x_0 \in A$, $y_0 = Sx_0 \in B$, $x_1 = Ty_0 \in A$ and $y_1 = Sx_1 \in B$ inductively, we have two sequences $\{x_n\}$ in A and $\{y_n\}$ in B such that

$$x_{n+1} = Ty_n \text{ and } y_n = Sx_n \text{ for } n \in \mathbb{N} \cup \{0\}. \quad (2.1)$$

Theorem 2.1. Let A and B be nonempty subsets of a cone metric space (X, d) and (S, T) a cone semi-cyclic φ -contraction pair.

Then for the sequences $\{x_n\}$ and $\{y_n\}$ generated as (2.1), two new sequences $\{d(x_n, Sx_n)\}$ and $\{d(y_{n+1}, Ty_n)\}$ are decreasing and converge to $d(A, B)$ in E .

Proof. For a strictly increasing mapping $\varphi : E \rightarrow E$

$$\varphi(d(A, B)) \preceq \varphi(d(x, y)) \text{ for all } x \in A \text{ and } y \in B,$$

so that

$$\begin{aligned} d(Sx, Ty) &\preceq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)) \\ &\preceq d(x, y). \end{aligned}$$

Thus we have

$$\begin{aligned} d(x_n, Sx_n) &= d(Ty_{n-1}, Sx_n) \\ &\preceq d(y_{n-1}, x_n) \\ &= d(Sx_{n-1}, Ty_{n-1}) \\ &\preceq d(x_{n-1}, y_{n-1}) \\ &= d(x_{n-1}, Sx_{n-1}). \end{aligned}$$

Hence the sequence $\{d(x_n, Sx_n)\}$ is decreasing in E .

Since $\{d(x_n, Sx_n)\}$ is bounded below, $\lim_{n \rightarrow \infty} d(x_n, Sx_n)$ exists.

Put $t_0 = \lim_{n \rightarrow \infty} d(x_n, Sx_n)$, then for $c \in \text{int}P$, there exists a natural number N such that if $n \geq N$, then

$$c - d(d(x_n, Sx_n), t_0) \in \text{int}P. \quad (2.2)$$

If $t_0 > d(A, B)$, then for $t_0 - d(A, B) \in P$, there exists $(x_0, y_0) \in A \times B$ such that

$$c - d(d(x_n, Sx_n), t_0) + d(A, B) - d(x_0, y_0) \in P. \quad (2.3)$$

Consequently, from (2.2) and (2.3), we have $d(A, B) - d(x_0, y_0) \in P \setminus \text{int}P \subset P$, which is a contradiction.

Hence we have $\lim_{n \rightarrow \infty} d(x_n, Sx_n) = d(A, B)$.

The same method also shows that $\{d(y_{n+1}, Ty_n)\}$ is decreasing and $\lim_{n \rightarrow \infty} d(y_{n+1}, Ty_n) = d(A, B)$. $\square$

Corollary 2.2. For a sequence $\{x_n\}$ constructed as $x_{n+1} = Tx_n$ for $x_0 \in A$ ($n \in \mathbb{N} \cup \{0\}$), where $T : A \cup B \rightarrow A \cup B$ is a cone cyclic φ -contraction, we have the same result as $d(x_n, x_{n+1}) \downarrow d(A, B)$.

Corollary 2.3. Let A and B be nonempty subsets of a cone metric space (M, d) . Let $T : A \cup B \rightarrow A \cup B$ be a cone cyclic k -contraction. Then $d(x_n, x_{n+1})$ converges $d(A, B)$, where x_0 is a given point of A and $x_{n+1} = Tx_n$ ($n \in \mathbb{N} \cup \{0\}$).

Proof. Now

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq k \cdot d(x_{n-1}, x_n) + (1 - k) \cdot d(A, B) \\ &\preceq k \cdot (k \cdot d(x_{n-2}, x_{n-1}) + (1 - k) \cdot d(A, B)) \\ &= k^2 \cdot d(x_{n-2}, x_{n-1}) + (1 - k^2) \cdot d(A, B) \\ &\preceq k^2 \cdot (k \cdot d(x_{n-3}, x_{n-2}) + (1 - k) \cdot d(A, B)) \\ &= k^3 \cdot d(x_{n-3}, x_{n-2}) + (1 - k^3) \cdot d(A, B). \end{aligned}$$

Inductively, we have

$$d(x_n, x_{n+1}) \preceq k^n \cdot d(x_0, x_1) + (1 - k^n) \cdot d(A, B).$$

Letting $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = d(A, B)$. $\square$

We have the corresponding result to Theorem 2.1 as a corollary in metric spaces as follows;

Corollary 2.4. [1]. Let A and B be nonempty subsets of a metric space X and $T : A \cup B \rightarrow A \cup B$ be a cyclic φ -contraction mapping. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$, then $d(x_n, x_{n+1}) \rightarrow d(A, B)$ as $n \rightarrow \infty$. Now we show that the generated sequences $\{x_n\}$ and $\{y_n\}$ in (2.1) are bounded.

Theorem 2.5. Let A and B be nonempty subsets of a normal cone metric space (X, d) and (S, T) be a cone semi-cyclic φ -contraction pair. For given point $x_0 \in A$, the sequences $\{x_n\}$ and $\{y_n\}$ generated as (2.1) are bounded.

Proof. We show that $\{x_n\}$ is bounded. Since $d(x_n, 0) \preceq d(x_n, Sx_n) + d(Sx_n, 0)$ and $\{d(x_n, Sx_n)\}$ is bounded, it is enough to show that $\{Sx_n\}$ is bounded. For the unbounded mapping φ , take $M \in E$ such that $\varphi(M) \succ d(x_0, x_1) + \varphi(d(A, B))$. If $\{Sx_n\}$ is not bounded, then there exists a natural number $N \in \mathbb{N}$ such that

$$d(x_1, Sx_N) \succ M \text{ and } d(x_1, Sx_{N-1}) \preceq M.$$

Hence by the definition of (S, T) ,

$$\begin{aligned}
 M \prec d(x_1, Sx_N) &= d(Ty_0, Sx_N) \\
 &\preceq d(y_0, x_N) \\
 &= d(Sx_0, Ty_{N-1}) \\
 &\preceq d(x_0, y_{N-1}) - \varphi(d(x_0, y_{N-1})) + \varphi(d(A, B)) \\
 &\preceq d(x_0, x_1) + d(x_1, Sx_{N-1}) - \varphi(d(x_0, Sx_{N-1})) + \varphi(d(A, B)) \\
 &\preceq d(x_0, x_1) + M - \varphi(d(x_0, Sx_{N-1})) + \varphi(d(A, B)),
 \end{aligned}$$

so,

$$\varphi(d(x_0, Sx_{N-1})) \prec d(x_0, x_1) + \varphi(d(A, B)). \quad (2.4)$$

On the other hand,

$$\begin{aligned}
 M \prec d(x_1, Sx_N) \\
 &\preceq d(x_0, y_{N-1}) \\
 &= d(x_0, Sx_{N-1}).
 \end{aligned} \quad (2.5)$$

Consequently, by (2.4) and (2.5)

$$\varphi(M) \prec \varphi(d(x_0, Sx_{N-1})) \prec d(x_0, x_1) + \varphi(d(A, B)) \prec \varphi(M),$$

which is a contradiction. Hence $\{x_n\}$ is bounded.

Similarly $\{y_n\}$ is bounded. □

3. Best Proximity Point

Now we show the best proximation result between two sets A and B in a normal cone metric space (X, d) for a cone semi-cyclic φ -contraction pair (S, T) .

Theorem 3.1. *Let A and B be nonempty subsets of a normal cone metric space (X, d) and (S, T) be a cone semi-cyclic φ -contraction pair. For a given point $x_0 \in A$, the sequences $\{x_n\}$ in A and $\{y_n\}$ in B generated as (2.1),*

- (i) *if $\{y_n\}$ has a convergent subsequence $\{y_{n_k}\}$ in B , then there exists a $y \in B$ such that $d(y, Ty) = d(A, B)$. In particular, if $d(A, B) = 0$, then the y is unique.*
- (ii) *if $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in A , then there exists a $x \in A$ such that $d(x, Sx) = d(A, B)$. In particular, if $d(A, B) = 0$, then the x is unique.*

Proof. (i) Let $y = \lim_{k \rightarrow \infty} y_{n_k}$. Since

$$\begin{aligned}
 d(A, B) &\preceq d(Ty_{n_k}, y) \\
 &\preceq d(Ty_{n_k}, y_{n_{k+1}}) + d(y_{n_{k+1}}, y),
 \end{aligned}$$

by letting $k \rightarrow \infty$, by Lemma 1.1 and Theorem 2.1 we have

$$\begin{aligned}
 d(A, B) &\preceq \lim_{k \rightarrow \infty} d(Ty_{n_k}, y) \\
 &\preceq \lim_{k \rightarrow \infty} d(Ty_{n_k}, y_{n_{k+1}}) + \lim_{k \rightarrow \infty} d(y_{n_{k+1}}, y). \\
 &= \lim_{k \rightarrow \infty} d(Ty_{n_k}, y_{n_{k+1}}) \\
 &= d(A, B).
 \end{aligned}$$

Hence

$$\begin{aligned}
 d(A, B) &\preceq \lim_{k \rightarrow \infty} d(Ty_{n_k}, y) \\
 &= d(A, B).
 \end{aligned}$$

Hence

$$d(A, B) = \lim_{k \rightarrow \infty} d(Ty_{n_k}, y). \quad (3.1)$$

On the other hand,

$$\begin{aligned}
 d(A, B) &\preceq d(Ty, y_{n_k}) \\
 &= d(Ty, Sx_{n_k}) \\
 &\preceq d(y, x_{n_k}) \\
 &= d(y, Ty_{n_k-1}).
 \end{aligned} \tag{3.2}$$

Letting $k \rightarrow \infty$ in (3.2), from (3.1), we have

$$\begin{aligned}
 d(A, B) &\preceq \lim_{k \rightarrow \infty} d(Ty, y_{n_k}) \\
 &\preceq \lim_{k \rightarrow \infty} d(y, Ty_{n_k-1}) \\
 &= d(A, B).
 \end{aligned}$$

Hence by Lemma 1.1,

$$\begin{aligned}
 d(A, B) &= \lim_{k \rightarrow \infty} d(Ty, y_{n_k}) \\
 &= d(Ty, y).
 \end{aligned}$$

On the other hand, assume that $d(y, Ty) = d(A, B)$, $d(z, Tz) = d(A, B)$ and $y \neq z$ in B . Since

$$\begin{aligned}
 d(y, z) &\preceq d(y, Ty) + d(Ty, z) \\
 &\preceq d(y, Ty) + d(Ty, Tz) + d(Tz, z), \\
 d(y, z) - d(Ty, Tz) &\preceq d(y, Ty) + d(Tz, z) \\
 &= 2d(A, B) \\
 &= 0,
 \end{aligned}$$

which shows that

$$\begin{aligned}
 d(y, z) &\preceq d(Ty, Tz) \\
 &\preceq d(y, z) - \varphi(d(y, z)) + \varphi(d(A, B)).
 \end{aligned}$$

Hence

$$\varphi(0) \prec \varphi(d(y, z)) \preceq \varphi(d(A, B)) = \varphi(0),$$

which is a contradiction.

(ii) It can be proved by the same method. □

Remark 3.1. If $A := \{x_0\}$ is a singleton, then $\{x_n\}_{n \in \mathbb{N}} = \{x_0\}$ and $\{y_n\}_{n \in \mathbb{N}} = \{y_0\} := \{Sx_0\} \in B$. Hence $d(x_0, y_0) = d(x_0, B) = \inf_{y \in B} d(x_0, y)$, which shows that B is closed. Similarly A is also closed.

REFERENCES

- [1] M. A. Al-Thagafi and N. Shahzad, *Convergence and existence results for best proximity points*, Nonlinear Analysis **70** (2009), 3665-3671.
- [2] A. A. Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. **232** (2006), 1001-1006.
- [3] M. Gabeleh and A. Abkar, *Best proximity points for semi-cyclic contractive pairs in Banach spaces*, Int. Math. Forum **6(44)** (2011), 2179-2186.
- [4] L.G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1468-1476.
- [5] D. Turkoglu, M. Abuloha and T. Abdeljawad, *KKM mappings in cone metric spaces and some fixed point theorems*, Nonlinear Analysis **72** (2010), 348-353.
- [6] V. Zabrejko, *K-metric and K-normed spaces: a survey*, Collect. Math. **48(4-6)** (1997), 825-859.