Star-based a Posteriori Error Estimator for Convection Diffusion Problems

B. Achchab¹, A. Agouzal^{2,*}, N. Debit², K. Bouihat¹

¹Université Hassan 1^{er}, LM²CE et LAMSAD, FSEJSS et ESTB, B.P 218 Berrechid, Morocco

²Université de Lyon; CNRS ; Université Lyon 1 ; Institut Camille Jordan,

Bâtiment Jean Braconnier 21, Avenue Claude Bernard 69622 Villeurbanne cedex, France

*Corresponding Author: agouzal@univ-lyon1.fr

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Abstract In this paper, we derive an a posteriori error estimator, for nonconforming finite element approximation of convection-diffusion equation. The a posteriori error estimator is based on the local problems on stars. Finally, we prove the reliability and the efficiency of the estimator without saturation assumption nor comparison with residual estimator

Keywords A posteriori error estimator, nonconforming finite elements method, convection diffusion equations

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1 Introduction

A posteriori error estimators provide the basis for adaptive mesh refinement and quantitative error control [1, 8, 4, 5, 12, 19, 14, 10]. One of the most successful estimators was proposed by Bank and Weiser and extended by many authors [2, 3, 7, 13, 16, 20, 21], it is based on the solution of local Neumann problems on elements, which seem to allow for cancellation and thus lead to better results than the residual estimators. The classical proof of equivalence with the energy error require the *saturation assumption* : this says that the solution can be approximated asymptotically better with quadratic than with linear finite elements. The saturation assumption is shown to be superfluous by Nochetto in [17]. However, removing this assumption requires comparison with residual estimators. More recently, a new a posteriori error estimators on stars was proposed in [15], and the proof of the equivalence with energy error it applies directly without reference to residual estimators.

In this paper, we extended the results of [15] to the case of nonconforming finite elements and the convection diffusion case. A new a posteriori error estimator is introduced based on the solution of a small discrete problem in stars. We prove the reliability and the efficiency of the estimator without saturation assumption nor comparison with residual estimator. We consider the simpler case of nonconforming approximations for convection diffusion problem, and we introduced a technique which allowed us to define a new a posteriori error estimator which are equivalent to the energy error.

2 Setting the problem

We consider here the convection-diffusion problem:

$$(P) \quad \begin{cases} -\varepsilon \Delta u + \beta \cdot \nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

In the following we assume that $\Omega \in \mathbf{R}^2$ a simply connected polygon domain, $0 < \varepsilon < 1$, $\beta \in (W^{1,\infty}(\Omega))^2$, such that $-\frac{1}{2} \operatorname{div} \beta \geq a > 0$ and $f \in L^2(\Omega)$. Let $\mathcal{T}_h$ be a family of conforming shape-regular triangulation of Ω by triangular, we denoted by E_I the set of interior edges and by E_f the set of all edges included in Γ . Let V_h be the

lowest order non-conforming Crouzeix-Raviart finite element space defined by:

$$\begin{aligned} V_h &= \{v_h \in L^2(\Omega); \forall T \in \mathcal{T}_h, v_h|_T \in P_1(T), \\ &\quad \forall E \in E_I, \int_E [v_h]_E d\sigma = 0 \text{ and } \forall E \in E_f, \\ &\quad \int_E v_h d\sigma = 0\}. \end{aligned}$$

where $[\cdot]_E$ denoted the jump of the function across E . For each $T \in \mathcal{T}_h$, we denote by $P_k(T)$ the polynomial space of degree less than or equal to k .

For all $T \in \mathcal{T}_h$, we define ∂T^- such the part of the frontier of T such that $\beta \cdot n_T < 0$ where n_T stands for the unit outward normal vector to T on ∂T .

In the sequel, we consider $u_h^{NC} \in V_h$ be a solution of the stabilized nonconforming approximation problem:

$$(P_h)^{NC} \begin{cases} \forall v_h \in V_h, \\ \sum_{T \in \mathcal{T}_h} \int_T [\varepsilon \nabla u_h^{NC} \cdot \nabla v_h + \beta \cdot \nabla u_h^{NC} v_h] dx \\ + \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{\partial T^-} \beta \cdot n [u_h^{NC}] v_h d\sigma = \int_{\Omega} f v_h dx. \end{cases}$$

2.0.1 The a posteriori error estimator

For the a posteriori error analysis of the considered approximation, we need to define some local spaces and problems. We denoted by $\{x_i\}_{i \in \mathcal{N}}$ the set of all nodes of the triangulation $\mathcal{T}_h$. For each $i \in \mathcal{N}$, ϕ_i denoted the canonical continuous piecewise linear basis function corresponding to x_i . The star ω_i is the interior relative to Ω of the support of ϕ_i , and h_i is the maximal size of the elements forming ω_i . Finally, Γ_i will denote the union of the sides touching x_i that are contained in Ω , and $\bar{\Gamma}_i$ will denote the union of the sides touching x_i that are contained in $\bar{\Omega}$.

For each star ω_i , $i \in \mathcal{N}$, if x_i we introduce the space $V(\omega_i)$ defined by

$$V(\omega_i) = \{v \in H_{loc}^1(\omega_i) : \int_{\omega_i} v \phi_i dx = 0\},$$

if x_i is an interior node,
and

$$V(\omega_i) = \{v \in H_{loc}^1(\omega_i) : v = 0 \text{ on } \partial\omega_i \cap \Gamma\},$$

if x_i is a boundary node.

We have the following result of ([15])

Proposition 1 *There exists a constant C , only depending on the minimum angle of the triangulation but independent of the star being considered, such that:*

$$\forall v \in V(\omega_i) \quad \|v\|_{0,\omega_i} \leq Ch_i \left(\int_{\omega_i} |\nabla v|^2 \phi_i dx \right)^{1/2}. \quad (1)$$

We define the finite dimensional local spaces $\mathcal{P}_0^2(\omega_i)$ as follows,

Definition 2 *For $i \in \mathcal{N}$, let $\mathcal{P}^2(\omega_i)$ denote the space of continuous piecewise quadratic functions on star ω_i that vanish on $\partial\omega_i$. The spaces $\mathcal{P}_0^2(\omega_i)$ is defined by $\mathcal{P}_0^2(\omega_i) = \mathcal{P}^2(\omega_i) \cap V(\omega_i)$.*

In the following we consider the energy norm:

$$\|u\|_{\varepsilon,\omega_i}^2 = \varepsilon \|\nabla u\|_{0,\omega_i}^2 + \|u\|_{0,\omega_i}^2.$$

Let $u_h^{NC} \in V_h$ be fixed and we denoted by $\nabla_h u_h$ the vector belonging to $(L^2(\Omega))^2$ defined by

$$\forall T \in \mathcal{T}_h; \quad \nabla_h v_h = \nabla v_h \quad \text{on } T.$$

For each $i \in \mathcal{N}$, we consider the local problems :

$$(P_1)_i \quad \begin{cases} \text{Find } \eta_i \in \mathcal{P}_0^2(\omega_i) \text{ such that } \forall \mu_i \in \mathcal{P}_0^2(\omega_i) \\ \int_{\omega_i} (\varepsilon \nabla \eta_i \cdot \nabla \mu_i) \phi_i dx = \int_{\omega_i} (\varepsilon \nabla_h u_h^{NC} \cdot \nabla \mu_i) \phi_i dx \\ + \int_{\omega_i} (\beta \cdot \nabla_h u_h^{NC} \mu_i) \phi_i dx \\ + \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{\partial T^-} \beta \cdot n [u_h^{NC}] \mu_i \phi_i d\sigma - \int_{\omega_i} f \mu_i \phi_i dx. \end{cases}$$

and

$$(P_2)_i \quad \begin{cases} \text{Find } \alpha_i \in \mathcal{P}_0^2(\omega_i) \text{ such that } \forall \mu_i \in \mathcal{P}_0^2(\omega_i) \\ \int_{\omega_i} (\varepsilon \nabla \alpha_i \cdot \nabla \mu_i) \phi_i dx = \int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl}(\mu_i \phi_i) dx. \end{cases}$$

Using Lax-Milgram theorem, we prove that the discrete problems have unique solution. The problems $(P_1)_i$ estimate the approximation error, but the problems $(P_2)_i$ estimate the consistency error of the used method.

Finally we set:

$$\forall i \in \mathcal{N}, \quad E_{1,i}^2(u_h^{NC}) = \int_{\omega_i} \varepsilon |\nabla \eta_i|^2 \phi_i dx,$$

and

$$\forall i \in \mathcal{N}, \quad E_{2,i}^2(u_h^{NC}) = \int_{\omega_i} \varepsilon |\nabla \alpha_i|^2 \phi_i dx.$$

2.0.2 Upper bound of the error

In this section we prove the one of main results of this paper. First, we prove the upper bound of the error without oscillation. As in ([6],[9],[11]). Recall that [18]:

Lemma 3 (*Discrete Poincar and Friedrichs inequalities*).

There is a positives constants C depending only on the minimum angle of $\mathcal{T}_h$ the Ω such that:

$$\forall v \in H_0^1(\Omega) + V_h, \quad \|v\|_{0,\Omega}^2 \leq C \left(\sum_{T \in \mathcal{T}_h} \|\nabla v\|_{0,\Omega}^2 \right). \quad (2)$$

We have the following global upper bond of the error:

Theorem 4 Let $u_h^{NC} \in V_h$ such that $(P_h)^{NC}$ holds. There is a positive constant C_1 depending only on the minimum angle of $\mathcal{T}_h$ such that

$$\begin{aligned} \|u - u_h^{NC}\|_{\varepsilon,\Omega} &\leq C_1 \left[\left(\sum_{i \in \mathcal{N}} (E_{1,i}^2(u_h^{NC}) + E_{2,i}^2(u_h^{NC})) \right)^{\frac{1}{2}} \right. \\ &\quad + \|\beta\|_{1,\infty} h_i \left(\sum_{i \in \mathcal{N}} E_{2,i}^2(u_h^{NC}) \right)^{\frac{1}{2}}, \\ &\quad + \sum_{i \in \mathcal{N}} (h_i \sum_{T \subset \omega_i} \|[u_h^{NC}]\|_{L^2(\partial T)} \\ &\quad + (\|\beta\|_{0,\infty} + \|\beta\|_{H(\text{div}, \omega_i)}) h_i \|u_h^{NC}\|_{0,\omega_i}) \\ &\quad \left. + \text{osc}(f) \right]. \end{aligned} \quad (3)$$

where $\text{osc}(f)$ is the data oscillations defined by: $\text{osc}(f) = \left(\sum_{i \in \mathcal{N}} \alpha_i^2 \|(f - f_i) \phi_i^{\frac{1}{2}}\|_{0,\omega_i}^2 \right)^{\frac{1}{2}}$.

where $f_i = \frac{\int_{\omega_i} f \phi_i dx}{\int_{\omega_i} \phi_i dx}$ and $\alpha_i = \min(1, \frac{h_i}{\sqrt{\varepsilon}})$.

Proof. Remark that using Helmholtz-decomposition, we have

$$\nabla_h u_h^{NC} - \nabla u = \nabla w + \text{Curl } \zeta, \quad (4)$$

with $w \in H_0^1(\Omega)$, $\zeta \in H^1(\Omega)$ and $\int_{\Omega} \nabla w \cdot \text{Curl } \zeta dx = 0$.

Let us remark also that the orthogonality implies the following error decomposition :

$$\varepsilon \|\nabla_h u_h^{NC} - \nabla u\|_{0,\Omega}^2 = \varepsilon \|\nabla w\|_{0,\Omega}^2 + \varepsilon \|\text{Curl } \zeta\|_{0,\Omega}^2.$$

and the following equalities :

$$\varepsilon \|\nabla w\|_{\Omega}^2 = \int_{\Omega} \varepsilon (\nabla_h u_h^{NC} - \nabla u) \cdot \nabla w dx. \quad (5)$$

and

$$\varepsilon \|\text{Curl } \zeta\|_{0,\Omega}^2 = \int_{\Omega} \varepsilon (\nabla_h u_h^{NC} - \nabla u) \cdot \text{Curl } \zeta dx. \quad (6)$$

The estimates of expressions in (5) and (6) will be established respectively in the Lemmas 7 and 8. As a main tool we use the following Lemmas.

Lemma 5 For each node $i \in \mathcal{N}$ there exists a unique operator $\Pi_i : V(\omega_i) \longrightarrow \mathcal{P}_0^2(\omega_i)$, such that for any $v \in V(\omega_i)$ the following conditions hold :

1. For all edge $E \subset \Gamma_i$, $\int_E (v - \Pi_i v) \phi_i d\sigma = 0$.
2. $\int_{\omega_i} (v - \Pi_i v) \phi_i dx = 0$
3. $\varepsilon \left(\int_{\omega_i} |\nabla \Pi_i v|^2 \phi_i \right)^{\frac{1}{2}} + \left(\int_{\omega_i} |\Pi_i v|^2 \phi_i \right)^{\frac{1}{2}} \leq C \left[\varepsilon \left(\int_{\omega_i} |\nabla v|^2 \phi_i \right)^{\frac{1}{2}} + \left(\int_{\omega_i} |v|^2 \phi_i \right)^{\frac{1}{2}} \right],$
4. $\left(\sum_{i \in \mathcal{N}} \alpha_i^{-2} \|(v - \Pi_i(v))(\phi_i)^{\frac{1}{2}}\|_{0,\omega_i}^2 \right)^{\frac{1}{2}} \leq C \|v\|_{\varepsilon,\Omega}$

where the constant C depends only on the minimum angles of $\mathcal{T}_h$.

The Lemma 5, is an adaptation of arguments given in [15], and so the proof will be skipped.

Lemma 6 For each node $i \in \mathcal{N}$, Π_i defined in the 5, functions $v \in V(\omega_i)$, $\zeta \in V(\omega_i)$ and $u_h^{NC} \in V_h$. We have

$$\int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \nabla ((\Pi_i v) \phi_i) dx = \int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \nabla (v \phi_i) dx, \quad (7)$$

and

$$\int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl} ((\Pi_i \zeta) \phi_i) dx = \int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \nabla (\zeta \phi_i) dx. \quad (8)$$

Proof. If we denoted by $[\frac{\partial u_h^{NC}}{\partial n_E}] \in P_0(E)$ the jump of the normal derivative across the edge E . By Green formula and by lemma 5, we have

$$\begin{aligned} & \int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \nabla ((\Pi_i v) \phi_i) dx \\ &= \sum_{E \subset \omega_i} \int_E \varepsilon [\frac{\partial u_h^{NC}}{\partial n_E}] (\Pi_i v) \phi_i dx \\ &= \sum_{E \subset \omega_i} \int_E \varepsilon [\frac{\partial u_h^{NC}}{\partial n_E}] v \phi_i dx = \int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \nabla (v \phi_i) dx. \end{aligned}$$

To prove the second equality, if we denoted by

$[\frac{\partial u_h^{NC}}{\partial \tau_E}] \in P_0(E)$ the jump of the tangential derivative across the edge E . We have by Green formula and lemma 5

$$\begin{aligned} & \int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl} ((\Pi_i \zeta) \phi_i) dx = \sum_{E \subset \omega_i} \int_E \varepsilon [\frac{\partial u_h^{NC}}{\partial \tau_E}] (\Pi_i \zeta) \phi_i dx \\ &= \sum_{E \subset \omega_i} \int_E \varepsilon [\frac{\partial u_h^{NC}}{\partial \tau_E}] \zeta \phi_i dx = \int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl} (\zeta \phi_i) dx. \end{aligned}$$

The following Lemma give an estimate of expression (5):

Lemma 7 For $w \in H_0^1(\Omega)$ defined in (4), there exists positives constants C^* , depending only on the minimum angle of $\mathcal{T}_h$ such that

$$\begin{aligned} \|w\|_{\varepsilon,\Omega} &\leq C^* \left[\left(\sum_{i \in \mathcal{N}} (E_{1,i}^2(u_h^{NC})) \right)^{\frac{1}{2}} \right. \\ &\quad + \|\beta\|_{1,\infty} h_i \left(\sum_{i \in \mathcal{N}} E_{2,i}^2(u_h^{NC}) \right)^{\frac{1}{2}}, \\ &\quad + \sum_{i \in \mathcal{N}} (h_i \sum_{T \subset \omega_i} \|[u_h^{NC}]\|_{L^2(\partial T)} + (\|\beta\|_{0,\infty} \\ &\quad + \|\beta\|_{H(\text{div}, \omega_i)} h_i \|u_h^{NC}\|_{0,\omega_i}, \\ &\quad \left. + \text{osc}(f) \right]. \end{aligned} \quad (9)$$

Proof . For $w \in H_0^1(\Omega)$ defined in (4), we set

$$w_h = \sum_{i \in \mathcal{N}} w_i \phi_i, \text{ where } w_i = \frac{\int_{\omega_i} w \phi_i dx}{\int_{\omega_i} \phi_i dx} \text{ for interior nodes, and } w_i = 0 \text{ otherwise.}$$

By adapting standard arguments used in the analysis of finite element approximation, and by using the propriety $-\frac{1}{2} \operatorname{div} \beta \geq a > 0$, we have:

$$\begin{aligned} a \|w\|_{\varepsilon, \Omega}^2 &\leq \int_{\Omega} \varepsilon \nabla w \cdot \nabla w \\ &\quad + \int_{\Omega} \beta \cdot \nabla w w dx, \end{aligned}$$

where: $\varepsilon \|\nabla w\|_{\Omega}^2 = \int_{\Omega} \varepsilon (\nabla_h u_h^{NC} - \nabla u) \cdot \nabla w dx$ and $\int_{\Omega} \nabla w \cdot \operatorname{Curl} \zeta dx = 0$.

This gives,

$$\begin{aligned} \int_{\Omega} \varepsilon \nabla w \cdot \nabla w + \int_{\Omega} \beta \cdot \nabla w w &= \int_{\Omega} \varepsilon \nabla w \cdot (\nabla_h u_h^{NC} - \nabla u) dx \\ &\quad + \int_{\Omega} \beta \cdot (\nabla_h u_h^{NC} - \nabla u) w \\ &\quad - \int_{\Omega} \beta \cdot w \operatorname{Curl} \zeta dx \\ &\quad - \int_{\Omega} \varepsilon \cdot \nabla w \cdot \operatorname{Curl} \zeta dx \\ &= \int_{\Omega} \varepsilon \nabla_h u_h^{NC} \cdot \nabla w dx \\ &\quad + \int_{\Omega} \beta \cdot \nabla_h u_h^{NC} w dx \\ &\quad - \int_{\Omega} \varepsilon \nabla u \cdot \nabla w dx - \int_{\Omega} \beta \cdot \nabla u w dx \\ &\quad - \int_{\Omega} \beta \cdot w \operatorname{Curl} \zeta dx \\ &\quad - \int_{\Omega} \varepsilon \cdot \nabla w \cdot \operatorname{Curl} \zeta dx. \end{aligned}$$

Since $w_h \in V_h \cap H_0^1(\Omega)$, $a(w, w_h) = 0$, we have: $a(w, w) = a(w, w - w_h)$. Then

$$\begin{aligned} a \|w\|_{\varepsilon, \Omega}^2 &\leq \int_{\Omega} \varepsilon \nabla_h u_h^{NC} \cdot \nabla (w - w_h) dx \\ &\quad + \int_{\Omega} \beta \cdot \nabla_h u_h^{NC} (w - w_h) dx \\ &\quad - \int_{\Omega} \varepsilon \nabla u \cdot \nabla (w - w_h) dx - \int_{\Omega} \beta \cdot \nabla u (w - w_h) dx \\ &\quad - \int_{\Omega} \beta \cdot (w - w_h) \operatorname{Curl} \zeta dx. \end{aligned}$$

Stating that

$w - w_h = \sum_{i \in \mathcal{N}} (w - w_i) \phi_i$, and using $\sum_{i \in \mathcal{N}} \phi_i = 1$ gives

$$\begin{aligned} a \|w\|_{\varepsilon, \Omega}^2 &\leq \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \nabla (w - w_i) \phi_i dx \right. \\ &\quad \left. + \int_{\omega_i} \beta \cdot \nabla_h u_h^{NC} (w - w_i) \phi_i dx \right] \\ &\quad - \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \varepsilon \nabla u \cdot \nabla (w - w_i) \phi_i dx \right. \\ &\quad \left. - \int_{\omega_i} \beta \cdot \nabla u (w - w_i) \phi_i dx \right] \\ &\quad - \int_{\omega_i} \beta \cdot (w - w_i) \phi_i \operatorname{Curl} \zeta dx. \end{aligned}$$

Since $(w - w_i) \in V(\omega_i)$, adding and removing same quantities in the two last terms give:

$$\begin{aligned}
 & a\|w\|_{\varepsilon,\Omega}^2 \\
 & \leq \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \nabla \Pi_i(w - w_i) \phi_i dx \right. \\
 & \quad + \int_{\omega_i} \beta \cdot \nabla_h u_h^{NC} \Pi_i(w - w_i) \phi_i dx \left. \right] \\
 & \quad + \sum_{i \in \mathcal{N}} \left[\frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] (\Pi_i(w - w_i)) \phi_i d\sigma \right] \\
 & \quad - \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} f \Pi_i(w - w_i) \phi_i dx \right] \\
 & \quad - \sum_{i \in \mathcal{N}} \left[\frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] (\Pi_i(w - w_i)) \phi_i d\sigma \right] \\
 & \quad + \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \beta \cdot \nabla_h u_h^{NC} (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\
 & \quad - \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} f (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\
 & \quad + \left[\int_{\omega_i} \beta \cdot (w - w_i) \phi_i \operatorname{Curl} \zeta dx \right].
 \end{aligned}$$

Using the definition of local problems $(P_1)_i$,

$$\begin{aligned}
 & a\|w\|_{\varepsilon,\Omega}^2 \\
 & \leq \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \varepsilon \nabla \eta_i \cdot \nabla \Pi_i(w - w_i) \phi_i dx \right] \\
 & \quad - \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} f (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\
 & \quad - \sum_{i \in \mathcal{N}} \left[\frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] (\Pi_i(w - w_i)) \phi_i d\sigma \right] \\
 & \quad + \left[\sum_{i \in \mathcal{N}} \int_{\omega_i} \beta \cdot \nabla_h u_h^{NC} (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\
 & \quad - \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \beta \cdot (w - w_i) \phi_i \operatorname{Curl} \zeta dx \right].
 \end{aligned}$$

We now process successively with each term of the right-hand side. On one hand, using Cauchy-Schwarz and item 3 of lemma 5 we have:

$$\begin{aligned}
 & \int_{\omega_i} \varepsilon \nabla \eta_i \cdot \nabla \Pi_i(w - w_i) \phi_i dx \\
 & \leq \left(\sum_{i \in \mathcal{N}} \int_{\omega_i} \varepsilon |\nabla \eta_i|^2 \phi_i dx \right)^{\frac{1}{2}} \left(\sum_{i \in \mathcal{N}} \int_{\omega_i} \varepsilon |\nabla \Pi_i(w - w_i)|^2 \phi_i dx \right)^{\frac{1}{2}}, \\
 & \leq C \left(\sum_{i \in \mathcal{N}} E_{1,i}^2(u_h^{NC}) \right)^{\frac{1}{2}} \left(\sum_{i \in \mathcal{N}} \int_{\omega_i} \varepsilon |\nabla(w - w_i)|^2 \phi_i dx \right)^{\frac{1}{2}}, \\
 & \leq C \left(\sum_{i \in \mathcal{N}} E_{1,i}^2(u_h^{NC}) \right)^{\frac{1}{2}} \|w\|_{\varepsilon,\Omega}.
 \end{aligned}$$

On the other hand, since both of $(w - w_i)$ and $\Pi_i(w - w_i)$ belong to $V(w_i)$, using definition of $V(w_i)$ and coefficients f_i give:

$$\begin{aligned}
 & \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} f (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\
 & = \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} (f - f_i) (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right]
 \end{aligned}$$

Using Cauchy-Schwarz, the proposition 1, the item 4 of lemma 5 and once more $\sum_{i \in \mathcal{N}} \phi_i = 1$ we get

$$\begin{aligned} & \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} f(w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\ & \leq \text{osc}(f) \left(\sum_{i \in \mathcal{N}} \alpha_i^{-2} \|(w - w_i - \Pi_i(w - w_i))(\phi_i)^{\frac{1}{2}}\|_{0, \omega_i}^2 \right)^{\frac{1}{2}}, \\ & \leq C \text{osc}(f) \|w\|_{\varepsilon, \Omega}. \end{aligned}$$

C is a generic constant only depending on the minimum angle of triangulation.

We note:

$$\begin{aligned} A &= \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \beta \cdot \nabla_h u_h^{NC} (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\ &\quad - \left[\frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] (\Pi_i(w - w_i)) \phi_i d\sigma \right] \end{aligned}$$

Using the Green formula we have:

$$\begin{aligned} A &= \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \beta \cdot \nabla_h u_h^{NC} (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\ &\quad - \left[\frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] (\Pi_i(w - w_i)) \phi_i d\sigma \right] \\ &= \sum_{i \in \mathcal{N}} \left[\frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] (w - w_i - \Pi_i(w - w_i)) \phi_i d\sigma \right] \\ &\quad - \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \beta \cdot u_h^{NC} \nabla (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\ &\quad + \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \text{div} \beta \cdot u_h^{NC} (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\ &\quad - \left[\frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] (\Pi_i(w - w_i)) \phi_i d\sigma \right]. \end{aligned}$$

Using Cauchy-Schwarz and the proposition 1 we have:

$$\begin{aligned} A &= \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \beta \cdot \nabla_h u_h^{NC} (w - w_i - \Pi_i(w - w_i)) \phi_i dx \right] \\ &\quad - \left[\frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] (\Pi_i(w - w_i)) \phi_i \right] \\ &\leq C \left[\sum_{i \in \mathcal{N}} (h_i \sum_{T \subset \omega_i} \| [u_h^{NC}] \|_{L^2(\partial T)} + (\|\beta\|_{0, \infty} \right. \\ &\quad \left. + \|\beta\|_{H(\text{div}, \omega_i)}) h_i \|u_h^{NC}\|_{0, \omega_i}) + \text{osc}(f) \right]. \end{aligned}$$

Finally:

$$\begin{aligned} & \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \beta \cdot (w - w_i) \text{Curl} \zeta dx \right] \\ & \leq \|\beta\|_{1, \infty} \|\text{Curl} \zeta\|_{0, \omega} \|w - w_i\|_{0, \omega} \\ & \leq C \|\beta\|_{1, \infty} \left(\sum_{i \in \mathcal{N}} E_{(2, i)}^2(u_h^{NC}) \right)^{\frac{1}{2}} \|w\|_{\varepsilon, \omega_i}. \end{aligned}$$

Summing up the different contributions in the estimate of $\|w\|_{\varepsilon, \omega_i}$ and using the continuity of $a(\cdot, \cdot)$ yield the result.

We have also the following result giving an estimation of expression (6):

Lemma 8 *For $\zeta \in H^1(\Omega)$ defined in (4), there exists a positive constant C , depending only on the minimum angle of $\mathcal{T}_h$ such that*

$$\varepsilon^{\frac{1}{2}} \|\text{Curl} \zeta\|_{0, \Omega} \leq C \left(\sum_{i \in \mathcal{N}} E_{(2, i)}^2(u_h^{NC}) \right)^{\frac{1}{2}}.$$

Proof. Let $\zeta \in H^1(\Omega)$ defined in (4), we note

$$\zeta_h = \sum_{i \in \mathcal{N}} \zeta_i \phi_i, \text{ where } \zeta_i = \frac{\int_{\omega_i} \zeta \phi_i dx}{\int_{\omega_i} \phi_i dx} \text{ for all nodes.}$$

First, using (6) and the fact that

$$\int_{\Omega} \varepsilon \nabla w \cdot \text{Curl} \zeta dx = 0.$$

Which easy to verify that:

$$\begin{aligned} \varepsilon \|\text{Curl} \zeta\|_{0,\Omega}^2 &= \int_{\Omega} \varepsilon (\nabla_h u_h^{NC} - \nabla u) \cdot \text{Curl} \zeta dx \\ &= \int_{\Omega} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl} \zeta dx, \end{aligned}$$

and hence:

$$\begin{aligned} \varepsilon \|\text{Curl} \zeta\|_{0,\Omega}^2 &= \int_{\Omega} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl} (\zeta - \zeta_h) dx \\ &\quad + \int_{\Omega} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl} \zeta_h dx. \end{aligned}$$

On one hand, since: $\text{Curl} \zeta_h \in H(\text{div}; \Omega)$, $\text{div}(\text{Curl} \zeta_h) = 0$, and $\forall T \in \mathcal{T}_h$, $(\text{Curl} \zeta_h)|_T$ belonging the lowest Raviart-Thomas space $RT_0(T) = (P_0(T))^2 + \underline{x} P_0(T)$ and $u_h^{NC} \in V_h$, by Green formula we have

$$\int_{\Omega} \nabla_h u_h^{NC} \cdot \text{Curl} \zeta_h dx = 0.$$

On the other hand, since $(\zeta - \zeta_i) \in V(\omega_i)$, using Lemma 5, and the definition of local problem $(P_2)_i$ we have, for all $i \in \mathcal{N}$:

$$\begin{aligned} &\sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl} ((\zeta - \zeta_i) \phi_i) dx \right] \\ &= \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl} (\Pi_i(\zeta - \zeta_i) \phi_i) dx \right] \\ &= \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \varepsilon (\nabla \alpha_i \cdot \nabla \Pi_i(\zeta - \zeta_i)) \phi_i dx \right]. \end{aligned}$$

Using the last equalities, $\sum_{i \in \mathcal{N}} \phi_i = 1$ and the equality $\|\text{Curl} \zeta\|_{0,\Omega} = \|\nabla \zeta\|_{0,\Omega}$, we obtain

$$\begin{aligned} &\varepsilon \|\text{Curl} \zeta\|_{0,\Omega}^2 \\ &= \sum_{i \in \mathcal{N}} \int_{\omega_i} \varepsilon (\nabla_h u_h^{NC} \cdot \text{Curl} (\zeta - \zeta_h)) \phi_i dx \\ &= \sum_{i \in \mathcal{N}} \int_{\omega_i} \varepsilon (\nabla_h u_h^{NC} \cdot \text{Curl} (\Pi_i(\zeta - \zeta_h))) \phi_i dx \\ &= \sum_{i \in \mathcal{N}} \int_{\omega_i} \varepsilon \nabla \alpha_i \cdot \nabla (\Pi_i(\zeta - \zeta_h)) \phi_i dx \\ &\leq C \left(\sum_{i \in \mathcal{N}} E_{(2,i)}^2(u_h^{NC}) \right)^{\frac{1}{2}} \left(\sum_{i \in \mathcal{N}} \int_{\omega_i} \varepsilon |\nabla \zeta|^2 \phi_i dx \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{i \in \mathcal{N}} E_{(2,i)}^2(u_h^{NC}) \right)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \|\text{Curl} \zeta\|_{0,\Omega}. \end{aligned}$$

Which completes proof of the lemma.

The combination of the two lemmas 7 and 8, gives the result of the upper bound of the error of the theorem 4.

2.0.3 Lower bound of the error

In this section we prove a lower bound of the error without oscillation.

Theorem 9 Let $u_h^{NC} \in V_h$, there exists a positives constants C_1 and C_2 , depending on the minimum angle of the triangulation such that, for any $i \in \mathcal{N}$,

$$E_{1,i}(u_h^{NC}) \leq C_1(1 + \|\beta\|_{1,\infty} h_i) \|u - u_h^{NC}\|_{\varepsilon, \omega_i},$$

and

$$E_{2,i}(u_h^{NC}) \leq C_2 \|u - u_h^{NC}\|_{\varepsilon, w_i}.$$

Proof. Let $i \in \mathcal{N}$. We have

$$\begin{aligned} E_{1,i}^2(u_h^{NC}) &= \int_{\omega_i} (\varepsilon \nabla_h u_h^{NC} \cdot \nabla \eta_i) \phi_i dx + \int_{\omega_i} (\beta \cdot \nabla_h u_h^{NC} \eta_i) \phi_i dx \\ &\quad - \int_{\omega_i} f \eta_i \phi_i dx + \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] \eta_i \phi_i d\sigma. \end{aligned}$$

Then

$$\begin{aligned} E_{(1,i)}^2(u_h) &= \int_{\omega_i} (\varepsilon \nabla_h u_h^{NC} \cdot \nabla \eta_i) \phi_i dx + \int_{\omega_i} (\beta \cdot \nabla_h u_h^{NC} \eta_i) \phi_i dx \\ &\quad - \int_{\omega_i} f \eta_i \phi_i dx \\ &\quad + \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] \eta_i \phi_i d\sigma, \\ &= \int_{\omega_i} (\varepsilon (\nabla_h u_h^{NC} - \nabla u) \cdot \nabla (\eta_i \phi_i)) dx \\ &\quad + \int_{\omega_i} (\beta \cdot (\nabla_h u_h^{NC} - \nabla u) \eta_i) \phi_i dx \\ &\quad + \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] \eta_i \phi_i d\sigma, \\ &= \int_{\omega_i} (\varepsilon (\nabla_h u_h^{NC} - \nabla u) \cdot \nabla (\eta_i) \phi_i) dx \\ &\quad + \int_{\omega_i} (\varepsilon (\nabla_h u_h^{NC} - \nabla u) \cdot \nabla (\phi_i) \eta_i) dx \\ &\quad + \int_{\omega_i} (\beta \cdot (\nabla_h u_h^{NC} - \nabla u) \eta_i) \phi_i dx \\ &\quad + \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] \eta_i \phi_i d\sigma. \end{aligned}$$

Using Cauchy-Schwarz theorem imply that:

$$\begin{aligned} E_{(1,i)}^2(u_h^{NC}) &\leq \varepsilon^{\frac{1}{2}} \|\nabla u - \nabla_h u_h^{NC}\|_{0, \omega_i} \left(\varepsilon \int_{\omega_i} |\nabla \eta_i|^2 \phi_i dx \right)^{\frac{1}{2}} \\ &\quad + \varepsilon^{\frac{1}{2}} \|\nabla u - \nabla_h u_h^{NC}\|_{0, \omega_i} \varepsilon^{\frac{1}{2}} \|\eta_i\|_{0, \omega_i} \|\phi_i\|_{1, \infty, \omega_i} \\ &\quad + \|\beta\|_{0, \infty} \|\nabla u - \nabla_h u_h^{NC}\|_{0, \omega_i} \left(\int_{\omega_i} |\eta_i|^2 \phi_i dx \right)^{\frac{1}{2}} \\ &\quad + C \| [u_h^{NC}] \|_{L^2(\partial T)} \|\eta_i\|_{0, \omega_i}. \end{aligned}$$

Or:

$$\varepsilon^{\frac{1}{2}} \|\eta_i\|_0 \leq C h_i \varepsilon^{\frac{1}{2}} \left(\int_{\omega_i} (|\nabla \eta_i|^2 \phi_i dx) \right)^{\frac{1}{2}}.$$

Then

$$\varepsilon^{\frac{1}{2}} \|\eta_i\|_{0, \omega_i} \leq C h_i E_{(1,i)}(u_h^{NC}).$$

and since $\eta_i \in V(\omega_i)$, we have $|\phi_i|_{1, \infty, \omega_i} \leq \frac{C}{h_i}$.

And:

$$\begin{aligned} &\sum_{T \in \mathcal{T}_h} \int_{\partial T^-} |\beta \cdot n| [u_h^{NC}] \eta_i \phi_i d\sigma \\ &\leq C \| [u_h^{NC}] \|_{L^2(\partial T)} \|\eta_i\|_{0, \omega_i} \\ &\leq C \varepsilon^{-\frac{1}{4}} \alpha_T^{\frac{1}{2}} \| [u_h^{NC}] \|_{L^2(\partial T)} E_{1,i}(u_h^{NC}) \\ &\leq C \|u - u_h^{NC}\|_{\varepsilon, w_i} E_{1,i}(u_h^{NC}). \end{aligned}$$

Summing up the different contributions in the estimate of $E_{1,i}(u_h^{NC})$ we get:

$$E_{1,i}(u_h^{NC}) \leq C(1 + \|\beta\|_{1,\infty} h_i) \|u - u_h^{NC}\|_{\varepsilon, w_i}.$$

To prove the second inequality, let us remark that, using Green formula, we have :

$$\forall \mu_i \in \mathcal{P}_0^2(\omega_i); \quad \int_{\omega_i} \varepsilon \nabla u \cdot \text{Curl}(\mu_i \phi_i) dx = 0.$$

Then using Cauchy-Schwartz inequality we deduce

$$\begin{aligned} E_{(2,i)}^2(u_h^{NC}) &= \int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl}(\alpha_i \phi_i) dx \\ &= \int_{\omega_i} \varepsilon (\nabla_h u_h^{NC} - \nabla u) \cdot \text{Curl}(\alpha_i \phi_i) dx \\ &\leq \varepsilon^{\frac{1}{2}} \|\nabla u - \nabla_h u_h^{NC}\|_{0, w_i} \varepsilon^{\frac{1}{2}} \|\text{Curl}(\alpha_i \phi_i)\|_{0, w_i}. \end{aligned}$$

Where

$$\varepsilon^{\frac{1}{2}} \|\text{Curl}(\alpha_i \phi_i)\|_{0, w_i} \leq C E_{(2,i)}(u_h^{NC}).$$

Then

$$E_{(2,i)}(u_h^{NC}) \leq C \varepsilon^{\frac{1}{2}} \|\nabla u - \nabla_h u_h^{NC}\|_{0, w_i} \leq C \|u - u_h^{NC}\|_{\varepsilon, w_i}$$

3 Mesh adaption procedure and Numerical results

Example 1. We now consider, on the same domain $\Omega =]0, 1[\times]0, 1[$, the convection diffusion reaction problem:

$$-\epsilon \Delta u + \beta \cdot \nabla u = f.$$

with $u = 0$ on $\partial\Omega$ such that $\beta = [3, 2]^t$, $\epsilon = 0.01$, and $f = 1$

This solution, for small $\epsilon > 0$, exhibits boundary layers near the top ($y = 1$) and right ($x = 1$) boundaries. The adaptive finite element method correctly refines in these layers, yielding accurate solutions with a small number of unknowns (relative to uniform refinement). The meshes and the contour maps are omitted for brevity reasons (see Fig 1 and Fig 2).

The interesting point of this problem concerns the exactness of estimators as ϵ goes to zero

Figure 1. Adaptive mesh refinement using the error indicator.

Figure 2. The isovalues of the solution .

Figure 3. Decay of error indicator. The dashed line has slope of $-\frac{1}{2}$.

Example 2. We now consider, on the same domain $\Omega =]0, 1[\times]0, 1[$, the convection diffusion reaction problem:

$$-\epsilon \Delta u + \beta \cdot \nabla u = f.$$

With the source term f given by the exact solution,

$$u = xy(x-1)(y-1)e^{-100(x-0.5)^2 - 100(y-0.5)^2}$$

which presents sharp curvature in the vicinity of point $(0.5, 0.5)$, and we perform a nonconforming finite element discretization on it. Successive iterations of adaptive mesh are represented in Figure 4. Computed and Exact solution are given in Figure 5, where the scaling of the height is the same for both pictures.

Remarks.

1. We can write a general framework regroups the conforming and nonconforming approximations of the convection diffusion problem, just write the approximate problem as:

$$(P_h) \quad \begin{cases} \text{Find } u_h \in W_h, \text{ such that : } \forall v_h \in W_h, \\ \sum_{T \in \mathcal{T}_h} \int_T [\epsilon \nabla u_h \cdot \nabla v_h \\ + \beta \cdot \nabla u_h v_h] dx + \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{\partial T^-} \beta \cdot n [u_h] v_h d\sigma \\ = \int_{\Omega} f v_h dx. \end{cases}$$

with $W_h = V_h \cap H_0^1(\Omega)$ in the conforming case and $W_h = V_h$ in the non conforming case. The same arguments can be used to derive an a posteriori error estimate on stars with similar properties.

2. The three dimensional case can be made with some regularity assumptions of the domain Ω , and by adapting the used technics in the two dimensional case.

Figure 4. Adaptive mesh refinement using the error indicator.

Figure 5. The isovals of the solution .

3.1 Conclusion

In this work we analyzed an a posteriori error estimator for nonconforming convection diffusion problem, with the Helmholtz-Decomposition technics. These estimators are efficient and robust to respect to physical parameters of the problem.

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