Moser's lemma and the Darboux theorem

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Abstract In this paper we will give a proof of the classical Moser's lemma. Using it, we give a proof of the main Darboux theorem, which states that every point in a symplectic manifold has a neighborhood with Darboux coordinates.

Keywords Symplectic manifolds, Moser lemma, Darboux theorem

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We will study in this note a theorem which plays a central role in symplectic geometry namely the Darboux theorem : the symplectic manifolds (M, ω) of dimension $2m$ are locally isomorphic to $(\mathbb{R}^{2m}, \omega)$. More precisely, if (M, ω) is a symplectic manifold of dimension $2m$, then in the neighborhood of each point of M , there exist local coordinates $(x_1, \dots, x_{2m})$ such that :

$$\omega = \sum_{k=1}^m dx_k \wedge dx_{m+k}.$$

In particular, there is no local invariant in symplectic geometry, analogous to the curvature in Riemannian geometry. The classical proof given by Darboux that bears his name is by induction on the dimension of the variety (see on this subject the excellent book by Arnold [1]). We will give an overview in remark 3. Another proof, based on a result of Moser [4], was given by Weinstein [5]. We will give in the following a proof of the Moser's lemma and then use this lemma to give a proof as direct as possible to the Darboux's theorem.

Lemma 1 *Let $\{\omega_t\}$, $0 \leq t \leq 1$, be a family of symplectic forms, differentiable in t . Then, for all $p \in M$, there exists a neighborhood $\mathcal{U}$ of p and a function $g_t : \mathcal{U} \rightarrow \mathcal{U}$, such that : $g_0^* = \text{identity}$ et $g_t^* \omega_t = \omega_0$.*

Proof : Looking for a family of vector fields X_t on $\mathcal{U}$ such that these fields generate locally a one-parameter group of diffeomorphisms g_t with

$$\frac{d}{dt} g_t(p) = X_t(g_t(p)), \quad g_0(p) = p.$$

First note that the form ω_t is closed (i.e., $d\omega_t = 0$) as $\frac{d}{dt} \omega_t$ since

$$d \frac{d}{dt} \omega_t = \frac{d}{dt} d\omega_t = 0.$$

Therefore, by deriving the relationship $g_t^* \omega_t = \omega_0$ and using the Cartan homotopy formula

$$L_{X_t} = i_{X_t} d + di_{X_t},$$

taking into account that ω_t depends on time, we obtain the expression

$$\frac{d}{dt} g_t^* \omega_t = g_t^* \left(\frac{d}{dt} \omega_t + L_{X_t} \omega_t \right) = g_t^* \left(\frac{d}{dt} \omega_t + di_{X_t} \omega_t \right).$$

By Poincaré's lemma (in the neighborhood of a point, any closed differential form is exact), the form $\frac{\partial}{\partial t} \omega_t$ is exact in the neighborhood of p . In other words, we can find a form λ_t such that :

$$\frac{d}{dt} \omega_t = d\lambda_t.$$

Hence

$$\frac{d}{dt}g_t^*\omega_t = g_t^*d(\lambda_t + i_{X_t}\omega_t). \quad (1)$$

We want to show that for all $p \in M$, there exists a neighborhood $\mathcal{U}$ of p and a function $g_t : \mathcal{U} \rightarrow \mathcal{U}$, such that : $g_0^* = \text{identity}$ and $g_t^*\omega_t = \omega_0$, therefore such that :

$$\frac{d}{dt}g_t^*\omega_t = 0.$$

And by (1), the problem amounts to finding X_t such that :

$$\lambda_t + i_{X_t}\omega_t = 0.$$

Since the form ω_t is nondegenerate, then the above equation is solvable with respect to the vector field X_t and defines the family $\{g_t\}$ pour $0 \leq t \leq 1$. In local coordinates (x_k) of the variety M of dimension $2m$, with $\left(\frac{\partial}{\partial x_k}\right)$ a basis of TM and (dx_k) the dual basis of $\left(\frac{\partial}{\partial x_k}\right)$, $k = 1, \dots, 2m$, we have

$$\begin{aligned} \lambda_t &= \sum_{k=1}^{2m} \lambda_k(t, x) dx_k, \\ X_t &= \sum_{k=1}^{2m} X_k(t, x) \frac{\partial}{\partial x_k}, \\ \omega_t &= \sum_{\substack{k,l=1 \\ k < l}}^{2m} \omega_{k,l}(t, x) dx_k \wedge dx_l, \\ i_{X_t}\omega_t &= 2 \sum_{l=1}^{2m} \left(\sum_{k=1}^{2m} \omega_{k,l} X_k \right) dx_l. \end{aligned}$$

We must therefore solve the system of equations in $x_k(t, x)$ according to :

$$\lambda_l(t, x) + 2 \sum_{k=1}^{2m} \omega_{k,l}(t, x) X_k(t, x) = 0.$$

The form ω_t is nondegenerate, then the matrix $(\omega_{k,l}(t, x))$ is nonsingular and therefore the above system has a unique solution. This determines the vector field X_t and thus functions g_t^* such that : $g_t^*\omega_t = \omega_0$, which completes the proof. ■

We are now able to give an easy direct proof of the Darboux's theorem.

Theorem 2 Any symplectic form on a manifold M of dimension $2m$ is locally diffeomorphic to the standard form on $\mathbb{R}^{2m}$. In other words, if (M, ω) is a symplectic manifold of dimension $2m$, then in the neighborhood of each point of M , there exist local coordinates $(x_1, \dots, x_{2m})$ such that :

$$\omega = \sum_{k=1}^m dx_k \wedge dx_{m+k}.$$

Poof : Let $\{\omega_t\}$, $0 \leq t \leq 1$, be a family of 2-differential forms which depends differentiably of t and let

$$\begin{aligned} \omega_t &= \omega_0 + t(\omega - \omega_0), \\ \omega_0 &= \sum_{k=1}^m dx_k \wedge dx_{m+k}, \end{aligned}$$

where $(x_1, \dots, x_{2m})$ are local coordinates on M . Note that these 2-forms are closed. At $p \in M$, we have

$$\omega_t(p) = \omega_0(p) = \omega(p).$$

By continuity, we can find a small neighborhood of p where the form $\omega_t(p)$ is nondegenerate. So the 2-forms ω_t are nondegenerate in a neighborhood of p and independent of t at p . In other words, ω_t are symplectic forms and by Moser's lemma, for all $p \in M$, there exists a neighborhood $\mathcal{U}$ of p and a function $g_t : \mathcal{U} \rightarrow \mathcal{U}$ such that : $g_t^* = \text{identité}$ and $g_t^*\omega_t = \omega_0$. Differentiating this relation with respect to t , we obtain (as in the proof of Moser's lemma),

$$\begin{aligned} \frac{d}{dt}g_t^*\omega_t &= 0, \\ g_t^* \left(\frac{d}{dt}\omega_t + L_{X_t}\omega_t \right) &= 0, \\ g_t^* \left(\frac{d}{dt}\omega_t + di_{X_t}\omega_t \right) &= 0. \end{aligned}$$

Therefore,

$$di_{X_t}\omega_t = -\frac{d}{dt}\omega_t,$$

and since the form $\frac{d}{dt}\omega_t$ is exact in the neighborhood of p (Poincaré's lemma), then

$$di_{X_t}\omega_t = d\theta_t,$$

where θ_t is a 1-differential form. In addition, ω_t being nondegenerate, the equation

$$i_{X_t}\omega_t = \theta_t,$$

is solvable and determines uniquely the vector field X_t depending on t . Note that for $t = 1$, $\omega_1 = \omega$ and for $t = 0$, $\omega_0 = \omega_0$ and also we can find g_1^* such that : $g_1^*\omega = \omega_0$. Vector fields X_t generate one-parameter families of diffeomorphisms $\{g_t\}$, $0 \leq t \leq 1$. In other words, you can make a change of coordinates as :

$$\omega = \sum_{k=1}^m dx_k \wedge d_{m+k},$$

and the proof is completed. ■

Remark 3 As we reported earlier this note, we'll provide an overview of the classical proof given by Darboux. We proceed by induction on m . Suppose the result true for $m-1 \geq 0$ and show that it is also for m . Fix x and let x_{m+1} be a differentiable function on M whose differential dx_m is a nonzero point x . Let X be the unique differentiable vector field satisfying the relation $i_X\omega = dx_{m+1}$. As this vector field does not vanish at x , then we can find a function x_1 in a neighborhood $\mathcal{U}$ of x such that $X(x_1) = 1$. Consider a vector field Y on $\mathcal{U}$ satisfying the relation $i_Y\omega = -dx_1$. Since $d\omega = 0$, then $L_X\omega = L_Y\omega = 0$, according to the Cartan homotopy formula. therefore

$$i_{[X,Y]}\omega = L_X i_Y\omega - i_Y(L_X\omega) = L_X(-dx_1) = -d(X(x_1)) = 0,$$

from which $[X, Y] = 0$, since any point in the form ω is of rank equal to $2m$. By the Recovery Theorem¹, it follows that there exist local coordinates $x_1, x_{m+1}, z_1, z_2, \dots, z_{2m-2}$ on a neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of x such that :

$$X = \frac{\partial}{\partial x_1}, \quad Y = \frac{\partial}{\partial x_{m+1}}.$$

Now consider the differential form

$$\lambda = \omega - dx_1 \wedge dx_{m+1}.$$

We have $d\lambda = 0$ and

$$i_X\lambda = L_X\lambda = i_Y\lambda = L_Y\lambda = 0.$$

So λ is expressed as a 2-differential form based only on variables $z_1, z_2, \dots, z_{2m-2}$. In particular, we have $\lambda^{m+1} = 0$. Furthermore, we have

$$0 \neq \omega^m = m dx_1 \wedge dx_{m+1} \wedge \lambda^{m-1}.$$

The 2-form λ is closed and of maximal rank (rank half) $m-1$ on an open set of $\mathbb{R}^{2m-2}$. It is therefore sufficient to apply the induction hypothesis to λ .

Remark 4 Suppose that the variety M is compact and connected. we show if

$$\int_M \omega_t = \int_M \omega_0,$$

where $\{\omega_t\}$, $0 \leq t \leq 1$ is a family of volume forms, then one can find a family of diffeomorphisms $g_t : M \rightarrow M$, such that : $g_0^* = \text{identity}$ and $g_t^*\omega_t = \omega_0$. Indeed, just use a reasoning similar to Moser's lemma, provided to replace the Poincaré's lemma which is local, by the De Rham's theorem which is global. This means that a volume form ω on M is exact if and only if $\int_M \omega = 0$.

1. Let $X_1, \dots, X_r$ differentiable vector fields on a manifold M and $x \in M$. Assume that for all $k, l = 1, \dots, r$, $[x_k, X_l] = 0$ and $X_1(x), \dots, X_r(x)$ are linearly independent. We show that there is an open $\mathcal{U}$ of M containing x and a local coordinate system on $\mathcal{U}$ such that :

$$X_1|_{\mathcal{U}} = \frac{\partial}{\partial x_1}, \dots, X_r|_{\mathcal{U}} = \frac{\partial}{\partial x_r}.$$

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