(i, j) - β - I - i -Open Sets and (i, j) - β - I - i -Almost Continuous Functions in Ideal Bitopological Spaces

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Abstract In this paper we introduce and study the notion of (i, j) - β - I - i -open sets in ideal bitopological spaces and investigate some of their properties. Also we present and study (i, j) - β - I - i -continuous and (i, j) - β - I - i -almost continuous functions. Furthermore, we obtain basic properties and preservation theorems of (i, j) - β - I - i -continuous and (i, j) - β - I - i -almost continuous.

Keywords Ideal bitopological spaces, i -open sets, (i, j) - β - I -open sets, (i, j) - β - I - i -open sets, (i, j) - β - I - i -continuous, (i, j) - β - I - i -almost continuous.

1 Introduction

A triple (X, τ_1, τ_2) where X is a non-empty set and τ_1 and τ_2 are topologies on X is called a bitopological space. An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (1) $A \in I$ and $B \subseteq A$ implies $B \in I$ and (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$. An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I) . The subject of ideals in topological spaces has been introduced and studied by Kuratowski [3] and Vaidyanathaswamy [7]. Let (X, τ_1, τ_2) be a bitopological space and let I be an ideal of subsets of X . An ideal bitopological space is a bitopological space (X, τ_1, τ_2) with an ideal I on X and is denoted by (X, τ_1, τ_2, I) . A set operator $(.)_i^* : P(X) \rightarrow P(X)$ is called the local function [7] of A with respect to τ_i and I , is defined as follows for $A \subseteq X$, $A_i^*(\tau_i, I) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau_i(x)\}$, where $\tau_i(x) = \{U \in \tau_i | x \in U\}$. For every ideal bitopological space (X, τ_1, τ_2, I) , there exists a topology $\tau_i^*(I)$, finer than τ_i . Additionally, $\tau_i - Cl^*(A) = A \cup A_i^*$ defines a Kuratowski closure operator [3] for $\tau_i^*(I)$. Also, $\tau_i - Cl^*(A) \subseteq \tau_i - Cl(A)$ for any subset A of X .

2 Preliminaries

For a subset A of a bitopological space (X, τ_1, τ_2) , we denote the closure of A and the interior of A with respect to τ_i by $\tau_i - Cl(A)$ and $\tau_i - Int(A)$, respectively.

Definition 2.1 [6] A subset A of X is said to be ij -semi open if $A \subseteq \tau_j - Cl(\tau_i - Int(A))$.

Definition 2.2 [5] A subset A of X is said to be ij - β -open if $A \subseteq \tau_j - Cl(\tau_i - Int(\tau_j - Cl(A)))$.

Definition 2.3 [1] A subset A of X is said to be ij -regular open if $A = \tau_i - Int(\tau_j - Cl(A))$.

Definition 2.4 [4] A point x of X is said to be ij - δ -cluster point of A if $A \cap U \neq \emptyset$ for every ij -regular open set U containing x , the set of all ij - δ -cluster points of A is called ij - δ -closure of A and denoted by $ij - Cl_\delta(A)$. A subset A of X is said to be ij - δ -closed if ij - δ -cluster points of A is subset of A , the complement of ij - δ -closed set is ij - δ -open and ij - δ -interior of A is denoted by $ij - Int_\delta(A)$.

Definition 2.5 [2] A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be (i, j) - β - I -open if $A \subseteq \tau_j - Cl(\tau_i - Int(\tau_j - Cl^*(A)))$, where $i, j = 1, 2$ and $i \neq j$.

Definition 2.6 [2] In an ideal bitopological space (X, τ_1, τ_2, I) , $A \subseteq X$ is said to be (i, j) - β - I -closed if $X \setminus A$ is (i, j) - β - I -open in X , $i, j = 1, 2$ and $i \neq j$.

Theorem 2.7 [2] Arbitrary intersection of (i, j) - β - I -closed sets is always (i, j) - β - I -closed.

Definition 2.8 [2] A function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j) - β - I -continuous if the inverse image of every i -open set of Y is (i, j) - β - I -open in X , where $i \neq j, i, j = 1, 2$.

3 (i, j) - β - I - i -Open Sets

Definition 3.1 An i -open subset A of a space X is called (i, j) - β - I - i -open if for each $x \in A$, there exists an (i, j) - β - I -closed set F such that $x \in F \subseteq A$.

The family of all (i, j) - β -I-i-open subsets of ideal bitopological space (X, τ_1, τ_2, I) is denoted by (i, j) - β -I-i $O(X, \tau_1, \tau_2, I)$ or (i, j) - β -I-i $O(X)$.

Theorem 3.2 *A subset A of a space X is (i, j) - β -I-i-open if and only if A is i -open and it is a union of (i, j) - β -I-closed sets. That is, $A = \bigcup F_\alpha$ where A is i -open and F_α is an (i, j) - β -I-closed set for each α .*

Proof. Obvious.

Remark 3.3 *It is clear from the definition that every (i, j) - β -I-i-open subset of a space X is i -open but not conversely.*

Theorem 3.4 *Every (i, j) - β -I-i-open set is (i, j) - β -I-open.*

Proof. Let A be an (i, j) - β -I-i-open subset of X . Then, we have $A \subseteq \tau_j - Cl^*(A)$ and hence $A = \tau_i - Int(A) \subseteq \tau_i - Int(\tau_j - Cl^*(A)) \subseteq \tau_j - Cl(\tau_i - Int(\tau_j - Cl^*(A)))$

The following example shows that the converses of Theorem 3.4 is not true in general.

Example 3.5 *Let $X = \{1, 2, 3, 4\}$, $\tau_1 = \{X, \varphi, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$, $\tau_2 = \{X, \varphi\}$ and $I = \{\varphi, \{3\}, \{4\}, \{3, 4\}\}$. Then the set $\{2, 3\}$ is $(1, 2)$ - β -I-open but not $(1, 2)$ - β -I-1-open.*

Theorem 3.6 *Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of (i, j) - β -I-i-open sets in an ideal bitopological space X . Then $\bigcup\{A_\alpha : \alpha \in \Delta\}$ is also (i, j) - β -I-i-open.*

Proof. Since A_α is an (i, j) - β -I-i-open set for each α , then A_α is i -open and $\bigcup\{A_\alpha : \alpha \in \Delta\}$ is i -open, so for all $x \in A_\alpha$, there exists an (i, j) - β -I-closed set F such that $x \in F \subseteq A_\alpha$ this implies that $x \in F \subseteq A_\alpha \subseteq \bigcup\{A_\alpha : \alpha \in \Delta\}$, then $x \in F \subseteq \bigcup\{A_\alpha : \alpha \in \Delta\}$, and hence $\bigcup\{A_\alpha : \alpha \in \Delta\}$ is an (i, j) - β -I-i-open set.

Theorem 3.7 *The intersection of two (i, j) - β -I-i-open sets is (i, j) - β -I-i-open.*

Proof. Let A and B be two (i, j) - β -I-i-open sets, then A and B are i -open sets this implies that $A \cap B$ is also an i -open set, we have to prove that $A \cap B$ is (i, j) - β -I-i-open, let $x \in A \cap B$ then $x \in A$ and $x \in B$, for all $x \in A$ there exists an (i, j) - β -I-closed set F such that $x \in F \subseteq A$ and for all $x \in B$ there exists an (i, j) - β -I-closed set E such that $x \in E \subseteq B$, and so that $x \in F \cap E \subseteq A \cap B$. Since the intersection of two (i, j) - β -I-closed sets is (i, j) - β -I-closed by Theorem 2.7, this shows that $A \cap B$ is (i, j) - β -I-i-open set.

From Theorems 3.6 and 3.7 we conclude that the family of all (i, j) - β -I-i-open subsets of a space X is a topology on X .

Theorem 3.8 *A subset A of a space (X, τ_1, τ_2, I) is (i, j) - β -I-i-open if and only if for each $x \in A$, there exists an (i, j) - β -I-i-open set B such that $x \in B \subseteq A$.*

Proof. Assume that A is an (i, j) - β -I-i-open set in (X, τ_1, τ_2, I) , let $x \in A$. If we put $B = A$ then B is an (i, j) - β -I-i-open set containing x such that $x \in B \subseteq A$. Conversely, suppose that for each $x \in A$, there exists an (i, j) - β -I-i-open set B_x such that $x \in B_x \subseteq A$, thus $A = \bigcup B_x$ where $B_x \in (i, j)$ - β -I-i $O(X)$ for each x , therefore A is (i, j) - β -I-i-open.

Definition 3.9 *A subset B of a space X is called (i, j) - β -I-i-closed if $X \setminus B$ is (i, j) - β -I-i-open.*

The family of all (i, j) - β -I-i-closed subsets of ideal bitopological space (X, τ_1, τ_2, I) is denoted by (i, j) - β -I-i $C(X, \tau_1, \tau_2, I)$ or (i, j) - β -I-i $C(X)$.

Theorem 3.10 *A subset B of a space X is (i, j) - β -I-i-closed if and only if B is an i -closed set and it is an intersection of (i, j) - β -I-open sets.*

Proof. Obvious.

Theorem 3.11 *Let $\{B_\alpha : \alpha \in \Delta\}$ be a collection of (i, j) - β -I-i-closed sets in an ideal bitopological space X . Then $\bigcap\{B_\alpha : \alpha \in \Delta\}$ is (i, j) - β -I-i-closed set.*

Proof. Follows from Theorem 3.6.

Theorem 3.12 *The union of two (i, j) - β -I-i-closed sets is (i, j) - β -I-i-closed.*

Proof. Follows from Theorem 3.7.

Definition 3.13 *Let (X, τ_1, τ_2, I) be an ideal bitopological space and $x \in X$. A subset N of X is said to be (i, j) - β -I-i-neighborhood of x if there exists an (i, j) - β -I-i-open set U in X such that $x \in U \subseteq N$.*

Theorem 3.14 *A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is (i, j) - β -I-i-open if and only if it is an (i, j) - β -I-i-neighborhood of each of its points.*

Proof. Let $A \subseteq X$ be an (i, j) - β -I-i-open set, since for every $x \in A$, $x \in A \subseteq A$ and A is (i, j) - β -I-i-open. This shows that A is (i, j) - β -I-i-neighborhood of each of its points.

Conversely, suppose that A is an (i, j) - β -I-i-neighborhood of each of its points, then for each $x \in A$, there exists $B_x \in (i, j)$ - β -I-i $O(X)$ such that $x \in B_x \subseteq A$. Therefore $A = \bigcup\{B_x : x \in A\}$. Since each B_x is (i, j) - β -I-i-open, it follows that A is an (i, j) - β -I-i-open set.

Definition 3.15 *Let A be a subset of an ideal bitopological space (X, τ_1, τ_2, I) . A point $x \in X$ is said to be (i, j) - β -I-i-interior point of A , if there exists an (i, j) - β -I-i-open set U such that $x \in U \subseteq A$. The set of all (i, j) - β -I-i-interior points of A is called (i, j) - β -I-i-interior of A and is denoted by (i, j) - β -I-i $Int(A)$.*

Theorem 3.16 *Let X be a space and $A \subseteq X$, $x \in X$. Then x is an (i, j) - β -I-i-interior point of A if and only if A is an (i, j) - β -I-i-neighborhood of x .*

Proof. Obvious.

Theorem 3.17 *Let A be any subset of a space X . If a point x is in the (i, j) - β -I-i-interior of A , then there exists an (i, j) - β -I-closed set F of X containing x such that $F \subseteq A$.*

Proof. Suppose that $x \in (i, j)$ - β -I-i $Int(A)$, then there exists an (i, j) - β -I-i-open set U of X containing x such that $U \subseteq A$. Since U is an (i, j) - β -I-i-open set, so there exists an (i, j) - β -I-closed set F containing x such that $x \in F \subseteq U \subseteq A$. Hence, $x \in F \subseteq A$.

Some properties of the (i, j) - β -I-i-interior of a set are investigated in the following Theorem.

Theorem 3.18 For subsets A and B of a space X , the following statements hold:

1. (i, j) - β - I - i $\text{Int}(A)$ is the union of all (i, j) - β - I - i open sets which are contained in A .
2. (i, j) - β - I - i $\text{Int}(A)$ is an (i, j) - β - I - i open set in X .
3. A is (i, j) - β - I - i open if and only if $A = (i, j)$ - β - I - i $\text{Int}(A)$.
4. (i, j) - β - I - i $\text{Int}((i, j)$ - β - I - i $\text{Int}(A)) = (i, j)$ - β - I - i $\text{Int}(A)$.
5. (i, j) - β - I - i $\text{Int}(\varphi) = \varphi$ and (i, j) - β - I - i $\text{Int}(X) = X$.
6. (i, j) - β - I - i $\text{Int}(A) \subseteq A$.
7. If $A \subseteq B$, then (i, j) - β - I - i $\text{Int}(A) \subseteq (i, j)$ - β - I - i $\text{Int}(B)$.
8. If $A \cap B = \varphi$, then (i, j) - β - I - i $\text{Int}(A) \cap (i, j)$ - β - I - i $\text{Int}(B) = \varphi$.
9. (i, j) - β - I - i $\text{Int}(A) \cup (i, j)$ - β - I - i $\text{Int}(B) \subseteq (i, j)$ - β - I - i $\text{Int}(A \cup B)$.
10. (i, j) - β - I - i $\text{Int}(A \cap B) = (i, j)$ - β - I - i $\text{Int}(A) \cap (i, j)$ - β - I - i $\text{Int}(B)$.
11. (i, j) - β - I - i $\text{Int}(A)$ is the largest (i, j) - β - I - i open set contained in A .

Proof. Obvious.

Definition 3.19 Let A be a subset of a space X . A point $x \in X$ is said to be (i, j) - β - I - i limit point of A if for each (i, j) - β - I - i open set U containing x , $U \cap (A \setminus \{x\}) \neq \varphi$.

The set of all (i, j) - β - I - i limit points of A is called the (i, j) - β - I - i derived set of A and is denoted by (i, j) - β - I - i $D(A)$.

Some properties of (i, j) - β - I - i derived set are stated in the following Theorem:

Theorem 3.20 Let A and B be subsets of a space X . Then we have the following properties:

1. (i, j) - β - I - i $D(\varphi) = \varphi$.
2. If $x \in (i, j)$ - β - I - i $D(A)$, then $x \in (i, j)$ - β - I - i $D(A \setminus \{x\})$.
3. If $A \subseteq B$, then (i, j) - β - I - i $D(A) \subseteq (i, j)$ - β - I - i $D(B)$.
4. (i, j) - β - I - i $D(A) \cup (i, j)$ - β - I - i $D(B) \subseteq (i, j)$ - β - I - i $D(A \cup B)$.
5. (i, j) - β - I - i $D(A \cap B) \subseteq (i, j)$ - β - I - i $D(A) \cap (i, j)$ - β - I - i $D(B)$.
6. (i, j) - β - I - i $D((i, j)$ - β - I - i $D(A)) \setminus A \subseteq (i, j)$ - β - I - i $D(A)$.
7. (i, j) - β - I - i $D(A \cup (i, j)$ - β - I - i $D(A)) \subseteq A \cup (i, j)$ - β - I - i $D(A)$.

Proof. We will prove (6) and (7) and the other parts can be proved obviously.

(6) $x \in (i, j)$ - β - I - i $D((i, j)$ - β - I - i $D(A)) \setminus A$ implies that $x \in (i, j)$ - β - I - i $D((i, j)$ - β - I - i $D(A))$ and U is an (i, j) - β - I - i open set containing x , then $U \cap ((i, j)$ - β - I - i $D(A) \setminus \{x\}) \neq \varphi$. Let $y \in U \cap ((i, j)$ - β - I - i $D(A) \setminus \{x\})$. Then, since $y \in (i, j)$ - β - I - i $D(A)$ and $y \in U$, $U \cap (A \setminus \{y\}) \neq \varphi$. Let $z \in U \cap (A \setminus \{y\})$. Then, $z \neq x$ for $z \in A$ and $x \notin A$. Hence, $U \cap (A \setminus \{x\}) \neq \varphi$. Therefore, $x \in (i, j)$ - β - I - i $D(A)$.

(7) Let $x \in (i, j)$ - β - I - i $D(A \cup (i, j)$ - β - I - i $D(A))$. If $x \in A$, the result is obvious. So, let $x \in (i, j)$ - β - I - i $D(A \cup (i, j)$ - β - I - i $D(A)) \setminus A$, then, for any (i, j) - β - I - i open set U containing x , $U \cap (A \cup (i, j)$ - β - I - i $D(A)) \setminus \{x\} \neq \varphi$. Thus, $U \cap (A \setminus \{x\}) \neq \varphi$ or $U \cap ((i, j)$ - β - I - i $D(A) \setminus \{x\}) \neq \varphi$. Now, it follows similarly from (6) that $U \cap (A \setminus \{x\}) \neq \varphi$. Hence, $x \in (i, j)$ - β - I - i $D(A)$. Therefore, in both cases, we obtain that (i, j) - β - I - i $D(A \cup (i, j)$ - β - I - i $D(A)) \subseteq A \cup (i, j)$ - β - I - i $D(A)$.

Definition 3.21 For any subset A in a space X , the (i, j) - β - I - i closure of A , denoted by (i, j) - β - I - i $Cl(A)$, is defined by the intersection of all (i, j) - β - I - i closed sets containing A .

Theorem 3.22 A subset A of an ideal bitopological space X is (i, j) - β - I - i closed if and only if it contains the set of its (i, j) - β - I - i limit points.

Proof. Assume that A is (i, j) - β - I - i closed and if possible suppose that x is an (i, j) - β - I - i limit point of A which belongs to $X \setminus A$, then $X \setminus A$ is (i, j) - β - I - i open set containing the (i, j) - β - I - i limit point x of A , therefore $A \cap X \setminus A \neq \varphi$, which is a contradiction.

Conversely, assume that A contains the set of its (i, j) - β - I - i limit points. For each $x \in X \setminus A$, there exists an (i, j) - β - I - i open set U containing x such that $A \cap U = \varphi$, that is, $x \in U \subseteq X \setminus A$ by Theorem 3.8, $X \setminus A$ is an (i, j) - β - I - i open set and hence, A is an (i, j) - β - I - i closed set.

Theorem 3.23 Let A be a set in a space X . A point $x \in X$ is in the (i, j) - β - I - i closure of A if and only if $A \cap U \neq \varphi$ for every (i, j) - β - I - i open set U containing x .

Proof. Let $x \notin (i, j)$ - β - I - i $Cl(A)$. Then $x \notin \bigcap F$, where F is (i, j) - β - I - i closed with $A \subseteq F$. So $x \in X \setminus \bigcap F$ and $X \setminus \bigcap F$ is an (i, j) - β - I - i open set containing x and hence, $(X \setminus \bigcap F) \cap A \subseteq (X \setminus \bigcap F) \cap (\bigcap F) = \varphi$. Conversely, suppose that there exists an (i, j) - β - I - i open set containing x with $A \cap U = \varphi$, then $A \subseteq X \setminus U$ and $X \setminus U$ is an (i, j) - β - I - i closed with $x \notin X \setminus U$. Hence, $x \notin (i, j)$ - β - I - i $Cl(A)$.

Theorem 3.24 Let A be any subset of a space X and x is a point of X . If $A \cap F \neq \varphi$ for every (i, j) - β - I - i closed set F of X containing x , then the point x is in the (i, j) - β - I - i closure of A .

Proof. Suppose that U is any (i, j) - β - I - i open set containing x , then by Definition 3.1, there exists an (i, j) - β - I - i closed set F such that $x \in F \subseteq U$. So by hypothesis $A \cap F \neq \varphi$ which implies that $A \cap U \neq \varphi$ for every (i, j) - β - I - i open set U containing x . Therefore, by Theorem 3.23, $x \in (i, j)$ - β - I - i $Cl(A)$.

Here, some properties of (i, j)- β -I-i-closure of the sets are introduced.

Theorem 3.25 For subsets A and B of a space X , the following statements are true:

1. The (i, j)- β -I-i-closure of A is the intersection of all (i, j)- β -I-i-closed sets containing A .
2. $A \subseteq (i, j)\text{-}\beta\text{-I-i } Cl(A)$.
3. (i, j)- β -I-i $Cl(A)$ is (i, j)- β -I-i-closed set in X .
4. A is (i, j)- β -I-i-closed set if and only if $A = (i, j)\text{-}\beta\text{-I-i } Cl(A)$.
5. (i, j)- β -I-i $Cl((i, j)\text{-}\beta\text{-I-i } Cl(A)) = (i, j)\text{-}\beta\text{-I-i } Cl(A)$.
6. (i, j)- β -I-i $Cl(\varphi) = \varphi$ and (i, j)- β -I-i $Cl(X) = X$.
7. If $A \subseteq B$, then (i, j)- β -I-i $Cl(A) \subseteq (i, j)\text{-}\beta\text{-I-i } Cl(B)$.
8. If (i, j)- β -I-i $Cl(A) \cap (i, j)\text{-}\beta\text{-I-i } Cl(B) = \varphi$, then $A \cap B = \varphi$.
9. (i, j)- β -I-i $Cl(A) \cup (i, j)\text{-}\beta\text{-I-i } Cl(B) = (i, j)\text{-}\beta\text{-I-i } Cl(A \cup B)$.
10. (i, j)- β -I-i $Cl(A \cap B) \subseteq (i, j)\text{-}\beta\text{-I-i } Cl(A) \cap (i, j)\text{-}\beta\text{-I-i } Cl(B)$.
11. (i, j)- β -I-i $Cl(A)$ is the smallest (i, j)- β -I-i-closed set containing A .

Proof. Obvious.

Theorem 3.26 For any subset A of an ideal bitopological space X , the following statements are true:

1. $X \setminus (i, j)\text{-}\beta\text{-I-i } Cl(A) = (i, j)\text{-}\beta\text{-I-i } Int(X \setminus A)$.
2. $X \setminus (i, j)\text{-}\beta\text{-I-i } Int(A) = (i, j)\text{-}\beta\text{-I-i } Cl(X \setminus A)$.
3. (i, j)- β -I-i $Cl(A) = X \setminus (i, j)\text{-}\beta\text{-I-i } Int(X \setminus A)$.
4. (i, j)- β -I-i $Int(A) = X \setminus (i, j)\text{-}\beta\text{-I-i } Cl(X \setminus A)$.

Proof. We prove (1) and the other parts can be proved similarly.

For any point $x \in X$, $x \in X \setminus (i, j)\text{-}\beta\text{-I-i } Cl(A)$ implies that $x \notin (i, j)\text{-}\beta\text{-I-i } Cl(A)$, then there exists $G \in (i, j)\text{-}\beta\text{-I-i } O(X)$ containing x , $A \cap G = \varphi$, then $x \in G \subseteq X \setminus A$. Thus, $x \in (i, j)\text{-}\beta\text{-I-i } Int(X \setminus A)$.

Conversely, the converse can be obtained by reversing the above steps.

Here, the (i, j)- β -I-i-boundary of a subset A is defined in a space X and study its properties.

Definition 3.27 Let A be a subset of a space X , then the (i, j)- β -I-i-boundary of A is defined as (i, j)- β -I-i $Cl(A) \setminus (i, j)\text{-}\beta\text{-I-i } Int(A)$ and is denoted by (i, j)- β -I-i $Bd(A)$.

Theorem 3.28 Let (X, τ_1, τ_2, I) be an ideal bitopological space and let A, B be subsets of X . Then

1. (i, j)- β -I-i $Bd(A) = (i, j)\text{-}\beta\text{-I-i } Cl(A) \cap (i, j)\text{-}\beta\text{-I-i } Cl(X \setminus A)$.

2. (i, j)- β -I-i $Bd(A)$ is an (i, j)- β -I-i-closed set.
3. (i, j)- β -I-i $Int(A) \cap (i, j)\text{-}\beta\text{-I-i } Bd(A) = \varphi$.
4. (i, j)- β -I-i $Cl(A) = (i, j)\text{-}\beta\text{-I-i } Int(A) \cup (i, j)\text{-}\beta\text{-I-i } Bd(A)$.
5. (i, j)- β -I-i $Bd((i, j)\text{-}\beta\text{-I-i } Int(A)) \subseteq (i, j)\text{-}\beta\text{-I-i } Bd(A)$.
6. (i, j)- β -I-i $Bd((i, j)\text{-}\beta\text{-I-i } Cl(A)) \subseteq (i, j)\text{-}\beta\text{-I-i } Bd(A)$.
7. (i, j)- β -I-i $Bd((i, j)\text{-}\beta\text{-I-i } Bd(A)) \subseteq (i, j)\text{-}\beta\text{-I-i } Bd(A)$.
8. (i, j)- β -I-i $Bd(A) = (i, j)\text{-}\beta\text{-I-i } Bd(X \setminus A)$.
9. (i, j)- β -I-i $Int(A) = A \setminus (i, j)\text{-}\beta\text{-I-i } Bd(A)$.
10. $X = (i, j)\text{-}\beta\text{-I-i } Int(A) \cup (i, j)\text{-}\beta\text{-I-i } Int(X \setminus A) \cup (i, j)\text{-}\beta\text{-I-i } Bd(A)$.
11. $X \setminus (i, j)\text{-}\beta\text{-I-i } Bd(A) = (i, j)\text{-}\beta\text{-I-i } Int(A) \cup (i, j)\text{-}\beta\text{-I-i } Int(X \setminus A)$.
12. (i, j)- β -I-i $Bd(A \cup B) \subseteq (i, j)\text{-}\beta\text{-I-i } Bd(A) \cup (i, j)\text{-}\beta\text{-I-i } Bd(B)$.
13. (i, j)- β -I-i $Bd(A \cap B) \subseteq (i, j)\text{-}\beta\text{-I-i } Bd(A) \cup (i, j)\text{-}\beta\text{-I-i } Bd(B)$.

Proof. we have:

1. (i, j)- β -I-i $Bd(A) = (i, j)\text{-}\beta\text{-I-i } Cl(A) \setminus (i, j)\text{-}\beta\text{-I-i } Int(A) = (i, j)\text{-}\beta\text{-I-i } Cl(A) \cap X \setminus (i, j)\text{-}\beta\text{-I-i } Int(A) = (i, j)\text{-}\beta\text{-I-i } Cl(A) \cap (i, j)\text{-}\beta\text{-I-i } Cl(X \setminus A)$.
2. (i, j)- β -I-i $Cl((i, j)\text{-}\beta\text{-I-i } Bd(A)) = (i, j)\text{-}\beta\text{-I-i } Cl((i, j)\text{-}\beta\text{-I-i } Cl(A) \cap (i, j)\text{-}\beta\text{-I-i } Cl(X \setminus A)) \subseteq (i, j)\text{-}\beta\text{-I-i } Cl((i, j)\text{-}\beta\text{-I-i } Cl(A)) \cap (i, j)\text{-}\beta\text{-I-i } Cl((i, j)\text{-}\beta\text{-I-i } Cl(X \setminus A)) = (i, j)\text{-}\beta\text{-I-i } Bd(A)$. Therefore, (i, j)- β -I-i $Bd(A)$ is (i, j)- β -I-i-closed.
3. (i, j)- β -I-i $Int(A) \cap (i, j)\text{-}\beta\text{-I-i } Bd(A) = (i, j)\text{-}\beta\text{-I-i } Int(A) \cap (i, j)\text{-}\beta\text{-I-i } Cl(A) \setminus (i, j)\text{-}\beta\text{-I-i } Int(A) = \varphi$.
4. (i, j)- β -I-i $Int(A) \cup (i, j)\text{-}\beta\text{-I-i } Bd(A) = (i, j)\text{-}\beta\text{-I-i } Int(A) \cup (i, j)\text{-}\beta\text{-I-i } Cl(A) \setminus (i, j)\text{-}\beta\text{-I-i } Int(A) = (i, j)\text{-}\beta\text{-I-i } Cl(A)$.
5. (i, j)- β -I-i $Bd((i, j)\text{-}\beta\text{-I-i } Int(A)) = (i, j)\text{-}\beta\text{-I-i } Cl((i, j)\text{-}\beta\text{-I-i } Int(A)) \setminus (i, j)\text{-}\beta\text{-I-i } Int((i, j)\text{-}\beta\text{-I-i } Int(A)) = (i, j)\text{-}\beta\text{-I-i } Cl((i, j)\text{-}\beta\text{-I-i } Int(A)) \setminus (i, j)\text{-}\beta\text{-I-i } Int(A) \subseteq (i, j)\text{-}\beta\text{-I-i } Cl(A) \setminus (i, j)\text{-}\beta\text{-I-i } Int(A) = (i, j)\text{-}\beta\text{-I-i } Bd(A)$.
6. (i, j)- β -I-i $Bd((i, j)\text{-}\beta\text{-I-i } Cl(A)) = (i, j)\text{-}\beta\text{-I-i } Cl((i, j)\text{-}\beta\text{-I-i } Cl(A)) \setminus (i, j)\text{-}\beta\text{-I-i } Int((i, j)\text{-}\beta\text{-I-i } Cl(A)) = (i, j)\text{-}\beta\text{-I-i } Cl(A) \setminus (i, j)\text{-}\beta\text{-I-i } Int((i, j)\text{-}\beta\text{-I-i } Cl(A)) \subseteq (i, j)\text{-}\beta\text{-I-i } Cl(A) \setminus (i, j)\text{-}\beta\text{-I-i } Int(A) = (i, j)\text{-}\beta\text{-I-i } Bd(A)$.
7. (i, j)- β -I-i $Bd((i, j)\text{-}\beta\text{-I-i } Bd(A)) = (i, j)\text{-}\beta\text{-I-i } Cl((i, j)\text{-}\beta\text{-I-i } Bd(A)) \cap (i, j)\text{-}\beta\text{-I-i } Cl(X \setminus (i, j)\text{-}\beta\text{-I-i } Bd(A)) \subseteq (i, j)\text{-}\beta\text{-I-i } Cl((i, j)\text{-}\beta\text{-I-i } Bd(A)) = (i, j)\text{-}\beta\text{-I-i } Bd(A)$, by (2).

8. $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus A) = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(X \setminus A)$.
9. $A \setminus ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \setminus ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A))) = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A)$.
10. Follows from (4) and Theorem 3.26(3).
11. Is a direct consequence of (10).
12. $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A \cup B) = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A \cup B) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus (A \cup B))$
 $= ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cup (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(B)) \cap ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus A) \cap (X \setminus B))$
 $\subseteq ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cup (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(B)) \cap ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus B))$
 $= \{(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus B))\} \cup \{(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(B) \cap ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus B))\}$
 $= \{(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus B)\} \cup \{(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(B) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus A)\}$
 $\subseteq (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \cup (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(B)$.
13. $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A \cap B) = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A \cap B) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus (A \cap B))$
 $\subseteq ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(B)) \cap ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus (A \cap B)))$
 $= ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(B)) \cap ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus A) \cup (X \setminus B))$
 $= ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(B)) \cap ((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus A) \cup (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus B))$
 $= \{((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(B)) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus A)\} \cup \{((i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(B)) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus B)\}$
 $= \{(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(B)\} \cup \{(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(B) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A)\}$
 $\subseteq (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \cup (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(B)$.

Next, $(i, j)\text{-}\beta\text{-}I\text{-}i$ -open and $(i, j)\text{-}\beta\text{-}I\text{-}i$ -closed sets in terms of $(i, j)\text{-}\beta\text{-}I\text{-}i$ -boundary are characterized.

Theorem 3.29 For a subset A of a space X , the following statements are true:

1. If A is $(i, j)\text{-}\beta\text{-}I\text{-}i$ -closed, then $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) = A \setminus (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A)$.
2. If A is $(i, j)\text{-}\beta\text{-}I\text{-}i$ -open, then $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \setminus A$.
3. If A is both $(i, j)\text{-}\beta\text{-}I\text{-}i$ -open and $(i, j)\text{-}\beta\text{-}I\text{-}i$ -closed, then $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) = \varphi$.
4. If A is $(i, j)\text{-}\beta\text{-}I\text{-}i$ -closed and $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A) = \varphi$, then $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) = A$.
5. A is $(i, j)\text{-}\beta\text{-}I\text{-}i$ -open if and only if $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } D(A)$.
6. A is $(i, j)\text{-}\beta\text{-}I\text{-}i$ -open if and only if $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \subseteq (X \setminus A)$. That is, $A \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) = \varphi$.
7. A is $(i, j)\text{-}\beta\text{-}I\text{-}i$ -closed if and only if $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \subseteq A$.

Proof. Only the proofs of (6) and (7) are given and the other proofs are obvious.

(6) Assume that $A \in (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } O(X)$, then by (2) $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \setminus A$. Hence, $A \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) = \varphi$. That is, $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \subseteq X \setminus A$.

Conversely, if $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \cap A = \varphi$, then by Theorem 3.28(1) $A \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap (X \setminus (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A)) = \varphi$. This implies that $A \cap (X \setminus (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A)) = \varphi$ by Theorem 3.26(2). Therefore, $A \subseteq X \setminus (X \setminus (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A)) = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A)$, but $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A) \subseteq A$ always. It follows that $A = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A)$ which implies that $A \in (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } O(X)$.

(7) Assume that $A \in (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } C(X)$, then by (1) $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) = A \setminus (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A) \subseteq A$.

Conversely, if $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \subseteq A$, then $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Bd(A) \cap X \setminus A = \varphi$ which implies that $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap (X \setminus (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Int(A)) \cap X \setminus A = \varphi$. Therefore, $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(X \setminus A)) \cap X \setminus A = \varphi$ which implies that $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \cap X \setminus A = \varphi$, and so $X \setminus A \subseteq X \setminus (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A)$. Thus, $(i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A) \subseteq A$, but $A \subseteq (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A)$ always. It follows that $A = (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } Cl(A)$ which implies that $A \in (i, j)\text{-}\beta\text{-}I\text{-}i \text{ } C(X)$.

The proof of others are simple, therefore omitted.

4 $(i, j)\text{-}\beta\text{-}I\text{-}i$ -Continuous and $(i, j)\text{-}\beta\text{-}I\text{-}i$ -Almost Continuous

Definition 4.1 A function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is called $(i, j)\text{-}\beta\text{-}I\text{-}i$ -continuous at a point $x \in X$ if for each i -open set V of Y containing $f(x)$, there exists an $(i, j)\text{-}\beta\text{-}I\text{-}i$ -open set U of X containing x such that $f(U) \subseteq V$. If f is $(i, j)\text{-}\beta\text{-}I\text{-}i$ -continuous at every point x of X , then it is called $(i, j)\text{-}\beta\text{-}I\text{-}i$ -continuous, where $i \neq j, i, j = 1, 2$.

Definition 4.2 A function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is called $(i, j)\text{-}\beta\text{-}I\text{-}i$ -almost continuous at a point $x \in X$ if for each i -open set V of Y containing $f(x)$, there exists an $(i, j)\text{-}\beta\text{-}I\text{-}i$ -open set U of X containing x such that $f(U) \subseteq \tau_i - Int(\tau_j - ClV)$. If f is $(i, j)\text{-}\beta\text{-}I\text{-}i$ -almost continuous at every point x of X , then it is called $(i, j)\text{-}\beta\text{-}I\text{-}i$ -almost continuous.

It is obvious from the definition that $(i, j)\text{-}\beta\text{-}I\text{-}i$ -continuous implies $(i, j)\text{-}\beta\text{-}I\text{-}i$ -almost continuous. However, the converse is not true in general as it is shown in the following example.

Example 4.3 Let $X = \{1, 2, 3\}$, $\tau_1 = \{X, \varphi, \{1\}\}$, $\tau_2 = \{X, \varphi, \{1\}, \{1, 2\}\}$, $\sigma_1 = \{X, \varphi, \{1, 3\}\}$, $\sigma_2 = \{X, \varphi, \{1\}\}$ and $I = \{\varphi, \{1\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, I) \rightarrow (X, \sigma_1, \sigma_2)$ is $(i, j)\text{-}\beta\text{-}I\text{-}i$ -almost continuous but not $(i, j)\text{-}\beta\text{-}I\text{-}i$ -continuous.

The proof of the following corollary follows directly from thier definitions.

Corollary 4.4

1. Every $(i, j)\text{-}\beta\text{-}I\text{-}i$ -continuous function is i -continuous.

2. Every (i, j)- β -I-i-continuous function is (i, j)- β -I-continuous.

Theorem 4.5 Let X and Y be bitopological spaces. A function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)- β -I-i-continuous if and only if the inverse image under f of every i -open set in Y is an (i, j)- β -I-i-open in X .

Proof. Assume that f is (i, j)- β -I-i-continuous and let V be any i -open set in Y . We have to show that $f^{-1}(V)$ is (i, j)- β -I-i-open in X . If $f^{-1}(V) = \varphi$, there is nothing to prove. So let $f^{-1}(V) \neq \varphi$ and let $x \in f^{-1}(V)$ so that $f(x) \in V$. By (i, j)- β -I-i-continuity of f , there exists an (i, j)- β -I-i-open set U in X containing x such that $f(U) \subseteq V$, that is $x \in U \subseteq f^{-1}(V)$, so $f^{-1}(V)$ is an (i, j)- β -I-i-open set.

Conversely, let $f^{-1}(V)$ be (i, j)- β -I-i-open in X for every i -open set V in Y . To show that f is (i, j)- β -I-i-continuous at $x \in X$, let V be any i -open set in Y such that $f(x) \in V$ so that $x \in f^{-1}(V)$. By hypothesis $f^{-1}(V)$ is (i, j)- β -I-i-open in X . If $f^{-1}(V) = U$, then U is an (i, j)- β -I-i-open set in X containing x such that

$$f(U) = f(f^{-1}(V)) \subseteq V$$

Hence f is an (i, j)- β -I-i-continuous function.

Theorem 4.6 For a function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

1. f is (i, j)- β -I-i-continuous.
2. $f^{-1}(V)$ is an (i, j)- β -I-i-open set in X , for each i -open set V in Y .
3. $f^{-1}(F)$ is an (i, j)- β -I-i-closed set in X , for each i -closed set F in Y .
4. $f((i, j)-\beta-I-i \text{ Cl}(A)) \subseteq \tau_i - \text{Cl}(f(A))$, for each subset A of X .
5. $(i, j)-\beta-I-i \text{ Cl}(f^{-1}(B)) \subseteq f^{-1}(\tau_i - \text{Cl}(B))$, for each subset B of Y .
6. $f^{-1}(\tau_i - \text{Int}(B)) \subseteq (i, j)-\beta-I-i \text{ Int}(f^{-1}(B))$, for each subset B of Y .
7. $\tau_i - \text{Int}(f(A)) \subseteq f((i, j)-\beta-I-i \text{ Int}(A))$, for each subset A of X .

Proof. (1) $\Rightarrow$ (2). Directly from Theorem 4.5.

(2) $\Rightarrow$ (3). Let F be any i -closed set of Y . Then $Y \setminus F$ is an i -open set of Y . By (2), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is (i, j)- β -I-i-open set in X and hence $f^{-1}(F)$ is (i, j)- β -I-i-closed set in X .

(3) $\Rightarrow$ (4). Let A be any subset of X . Then $f(A) \subseteq \tau_i - \text{Cl}(f(A))$ and $\tau_i - \text{Cl}(f(A))$ is i -closed in Y . Hence $A \subseteq f^{-1}(\tau_i - \text{Cl}(f(A)))$. By (3), we have $f^{-1}(\tau_i - \text{Cl}(f(A)))$ is an (i, j)- β -I-i-closed set in X . Therefore, $(i, j)-\beta-I-i \text{ Cl}(A) \subseteq f^{-1}(\tau_i - \text{Cl}(f(A)))$. Hence $f((i, j)-\beta-I-i \text{ Cl}(A)) \subseteq \tau_i - \text{Cl}(f(A))$.

(4) $\Rightarrow$ (5). Let B be any subset of Y . Then $f^{-1}(B)$ is a subset of X . By (4), we have $f((i, j)-\beta-I-i \text{ Cl}(f^{-1}(B))) \subseteq \tau_i - \text{Cl}(f(f^{-1}(B))) = \tau_i - \text{Cl}(B)$. Hence $(i, j)-\beta-I-i \text{ Cl}(f^{-1}(B)) \subseteq f^{-1}(\tau_i - \text{Cl}(B))$.

(5) $\Rightarrow$ (6). Let B be any subset of Y . Then apply (5) to $Y \setminus B$ is obtained $(i, j)-\beta-I-i \text{ Cl}(f^{-1}(Y \setminus B)) \subseteq$

$f^{-1}(\tau_i - \text{Cl}(Y \setminus B)) \Leftrightarrow (i, j)-\beta-I-i \text{ Cl}(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus \tau_i - \text{Int}(B)) \Leftrightarrow X \setminus (i, j)-\beta-I-i \text{ Int}(f^{-1}(B)) \subseteq X \setminus f^{-1}(\tau_i - \text{Int}(B)) \Leftrightarrow f^{-1}(\tau_i - \text{Int}(B)) \subseteq (i, j)-\beta-I-i \text{ Int}(f^{-1}(B))$. Therefore, $f^{-1}(\tau_i - \text{Int}(B)) \subseteq (i, j)-\beta-I-i \text{ Int}(f^{-1}(B))$.

(6) $\Rightarrow$ (7). Let A be any subset of X . Then $f(A)$ is a subset of Y . By (6), we have $f^{-1}(\tau_i - \text{Int}(f(A))) \subseteq (i, j)-\beta-I-i \text{ Int}(f^{-1}(f(A))) = (i, j)-\beta-I-i \text{ Int}(A)$. Therefore, $\tau_i - \text{Int}(f(A)) \subseteq f((i, j)-\beta-I-i \text{ Int}(A))$.

(7) $\Rightarrow$ (1). Let $x \in X$ and let V be any i -open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X . By (7), we have $\tau_i - \text{Int}(f(f^{-1}(V))) \subseteq f((i, j)-\beta-I-i \text{ Int}(f^{-1}(V)))$. Then $\tau_i - \text{Int}(V) \subseteq f((i, j)-\beta-I-i \text{ Int}(f^{-1}(V)))$. Since V is an i -open set. Then $V \subseteq f((i, j)-\beta-I-i \text{ Int}(f^{-1}(V)))$ implies that $f^{-1}(V) \subseteq (i, j)-\beta-I-i \text{ Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is an (i, j)- β -I-i-open set in X which contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence f is (i, j)- β -I-i-continuous.

Theorem 4.7 For a function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

1. f is (i, j)- β -I-i-almost continuous.
2. For each $x \in X$ and each ij -regular open set V of Y containing $f(x)$, there exists an (i, j)- β -I-i-open U in X containing x such that $f(U) \subseteq V$.
3. For each $x \in X$ and each ij - δ -open set V of Y containing $f(x)$, there exists an (i, j)- β -I-i-open U in X containing x such that $f(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and let V be any ij -regular open set of Y containing $f(x)$. By (1), there exists an (i, j)- β -I-i-open set U of X containing x such that $f(U) \subseteq \tau_i - \text{Int}(\tau_j - \text{Cl}(V))$. Since V is ij -regular open, then $\tau_i - \text{Int}(\tau_j - \text{Cl}(V)) = V$. Therefore, $f(U) \subseteq V$.

(2) $\Rightarrow$ (3). Let $x \in X$ and let V be any ij - δ -open set of Y containing $f(x)$. Then for each $f(x) \in V$, there exists an i -open set G containing $f(x)$ such that $G \subseteq \tau_i - \text{Int}(\tau_j - \text{Cl}(G)) \subseteq V$. Since $\tau_i - \text{Int}(\tau_j - \text{Cl}(G))$ is ij -regular open set of Y containing $f(x)$. By (2), there exists an (i, j)- β -I-i-open set U in X containing x such that $f(U) \subseteq \tau_i - \text{Int}(\tau_j - \text{Cl}(G)) \subseteq V$. This completes the proof.

(3) $\Rightarrow$ (1). Let $x \in X$ and let V be any i -open set of Y containing $f(x)$. Then $\tau_i - \text{Int}(\tau_j - \text{Cl}(V))$ is ij - δ -open set of Y containing $f(x)$. By (3), there exists an (i, j)- β -I-i-open set U in X containing x such that $f(U) \subseteq \tau_i - \text{Int}(\tau_j - \text{Cl}(V))$. Therefore, f is (i, j)- β -I-i-almost continuous.

Theorem 4.8 For a function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

1. f is (i, j)- β -I-i-almost continuous.
2. $f^{-1}(\tau_i - \text{Int}(\tau_j - \text{Cl}(V)))$ is an (i, j)- β -I-i-open set in X , for each i -open set V in Y .
3. $f^{-1}(\tau_i - \text{Cl}(\tau_j - \text{Int}(F)))$ is an (i, j)- β -I-i-closed set in X , for each i -closed set F in Y .
4. $f^{-1}(F)$ is an (i, j)- β -I-i-closed set in X , for each ij -regular closed set F of Y .

5. $f^{-1}(V)$ is an (i, j) - β - I - i -open set in X , for each ij -regular open set V of Y .
6. $f^{-1}(G)$ is an (i, j) - β - I - i -open set in X , for each ij - δ -open set G of Y .

Proof. (1) $\Rightarrow$ (2). Let V be any i -open set in Y . We have to show that $f^{-1}(\tau_i - \text{Int}(\tau_j - \text{Cl}(V)))$ is (i, j) - β - I - i -open set in X . Let $x \in f^{-1}(\tau_i - \text{Int}(\tau_j - \text{Cl}(V)))$. Then $f(x) \in \tau_i - \text{Int}(\tau_j - \text{Cl}(V))$ and $\tau_i - \text{Int}(\tau_j - \text{Cl}(V))$ is an ij -regular open set in Y . Since f is (i, j) - β - I -almost continuous, then by Theorem 4.7, there exists an (i, j) - β - I - i -open set U of X containing x such that $f(U) \subseteq \tau_i - \text{Int}(\tau_j - \text{Cl}(V))$. Which implies that $x \in U \subseteq f^{-1}(\tau_i - \text{Int}(\tau_j - \text{Cl}(V)))$. Therefore, $f^{-1}(\tau_i - \text{Int}(\tau_j - \text{Cl}(V)))$ is an (i, j) - β - I - i -open set in X .

(2) $\Rightarrow$ (3). Let F be any i -closed set of Y . Then $Y \setminus F$ is an i -open set of Y . By (2), $f^{-1}(\tau_i - \text{Int}(\tau_j - \text{Cl}(Y \setminus F)))$ is an (i, j) - β - I - i -open set in X and $f^{-1}(\tau_i - \text{Int}(\tau_j - \text{Cl}(Y \setminus F))) = f^{-1}(\tau_i - \text{Int}(Y \setminus \tau_j - \text{Int}(F))) = f^{-1}(Y \setminus \tau_i - \text{Cl}(\tau_j - \text{Int}(F))) = X \setminus f^{-1}(\tau_i - \text{Cl}(\tau_j - \text{Int}(F)))$ is an (i, j) - β - I - i -open set in X and hence $f^{-1}(\tau_i - \text{Cl}(\tau_j - \text{Int}(F)))$ is (i, j) - β - I - i -closed set in X .

(3) $\Rightarrow$ (4). Let F be any ij -regular closed set of Y . Then F is an i -closed set of Y . By (3), $f^{-1}(\tau_i - \text{Cl}(\tau_j - \text{Int}(F)))$ is (i, j) - β - I - i -closed set in X . Since F is ij -regular closed set, then $f^{-1}(\tau_i - \text{Cl}(\tau_j - \text{Int}(F))) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is an (i, j) - β - I - i -closed set in X .

(4) $\Rightarrow$ (5). Let V be any ij -regular open set of Y . Then $Y \setminus V$ is an ij -regular closed set of Y and by (4), we have $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is an (i, j) - β - I - i -closed set in X and hence $f^{-1}(V)$ is an (i, j) - β - I - i -open set in X .

(5) $\Rightarrow$ (6). Let G be any ij - δ -open set in Y , $G = \bigcup \{V_\alpha : \alpha \in \Delta\}$ where V_α is ij -regular open. Then $f^{-1}(G) = \bigcup \{f^{-1}(V_\alpha)\}$, from (5) we have $f^{-1}(V_\alpha)$ is an (i, j) - β - I - i -open set, then $f^{-1}(G) = \bigcup \{f^{-1}(V_\alpha)\}$ is an (i, j) - β - I - i -open.

(6) $\Rightarrow$ (1). Let $x \in X$ and let V be any ij - δ -open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$. By (6), we have $f^{-1}(V)$ is an (i, j) - β - I - i -open set in X . Therefore, we obtain $f(f^{-1}(V)) \subseteq V$. Hence by Theorem 4.7, f is (i, j) - β - I -almost continuous.

Theorem 4.9 For a function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

1. f is (i, j) - β - I -almost continuous.
2. $f((i, j)$ - β - I - i $\text{Cl}(A)) \subseteq ij - \text{Cl}_\delta(f(A))$, for each subset A of X .
3. (i, j) - β - I - i $\text{Cl}(f^{-1}(B)) \subseteq f^{-1}(ij - \text{Cl}_\delta(B))$, for each subset B of Y .
4. $f^{-1}(F)$ is (i, j) - β - I - i -closed set in X , for each ij - δ -closed set F of Y .
5. $f^{-1}(V)$ is (i, j) - β - I - i -open set in X , for each ij - δ -open set V of Y .
6. $f^{-1}(ij \text{Int}_\delta(B)) \subseteq (i, j)$ - β - I - i $\text{Int}(f^{-1}(B))$, for each subset B of Y .
7. $ij \text{Int}_\delta(f(A)) \subseteq f((i, j)$ - β - I - i $\text{Int}(A))$, for each subset A of X .

Proof. (1) $\Rightarrow$ (2). Let A be a subset of X . Since $ij - \text{Cl}_\delta(f(A))$ is an ij - δ -closed set in Y , then $Y \setminus ij - \text{Cl}_\delta(f(A))$ is ij - δ -open, from Theorem 4.8, $f^{-1}(Y \setminus ij - \text{Cl}_\delta(f(A)))$ is (i, j) - β - I - i -open, which implies that $X \setminus f^{-1}(ij - \text{Cl}_\delta(f(A)))$ is also (i, j) - β - I -open, so $f^{-1}(ij - \text{Cl}_\delta(f(A)))$ is (i, j) - β - I - i -closed set in X . Since $A \subseteq f^{-1}(ij - \text{Cl}_\delta(f(A)))$, so (i, j) - β - I - i $\text{Cl}(A) \subseteq f^{-1}(ij - \text{Cl}_\delta(f(A)))$. Therefore, $f((i, j)$ - β - I - i $\text{Cl}(A)) \subseteq ij - \text{Cl}_\delta(f(A))$.

(2) $\Rightarrow$ (3). Let B be a subset of Y . We have $f^{-1}(B)$ is a subset of X . By (2), we have $f((i, j)$ - β - I - i $\text{Cl}(f^{-1}(B))) \subseteq ij - \text{Cl}_\delta(f(f^{-1}(B))) = ij - \text{Cl}_\delta(B)$. Hence (i, j) - β - I - i $\text{Cl}(f^{-1}(B)) \subseteq f^{-1}(ij - \text{Cl}_\delta(B))$.

(3) $\Rightarrow$ (4). Let F be any ij - δ -closed set of Y . By (3), we have (i, j) - β - I - i $\text{Cl}(f^{-1}(F)) \subseteq f^{-1}(ij - \text{Cl}_\delta(F)) = f^{-1}(F)$ and hence $f^{-1}(F)$ is an (i, j) - β - I - i -closed set in X .

(4) $\Rightarrow$ (5). Let V be any ij - δ -open set of Y . Then $Y \setminus V$ is an ij - δ -closed set of Y and by (4), we have $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is an (i, j) - β - I - i -closed set in X . Hence $f^{-1}(V)$ is an (i, j) - β - I - i -open set in X .

(5) $\Rightarrow$ (6). For each subset B of Y . We have $ij \text{Int}_\delta(B) \subseteq B$. Then $f^{-1}(ij \text{Int}_\delta(B)) \subseteq f^{-1}(B)$. By (5), $f^{-1}(ij \text{Int}_\delta(B))$ is an (i, j) - β - I - i -open set in X . Then $f^{-1}(ij \text{Int}_\delta(B)) \subseteq (i, j)$ - β - I - i $\text{Int}(f^{-1}(B))$.

(6) $\Rightarrow$ (7). Let A be any subset of X . Then $f(A)$ is a subset of Y . By (6), we obtain that $f^{-1}(ij \text{Int}_\delta(f(A))) \subseteq (i, j)$ - β - I - i $\text{Int}(f^{-1}(f(A)))$. Hence $f^{-1}(ij \text{Int}_\delta(f(A))) \subseteq (i, j)$ - β - I - i $\text{Int}(A)$, which implies that $ij \text{Int}_\delta(f(A)) \subseteq f((i, j)$ - β - I - i $\text{Int}(A))$.

(7) $\Rightarrow$ (1). Let $x \in X$ and V be any ij -regular open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X . By (7), we get $ij \text{Int}_\delta(f(f^{-1}(V))) \subseteq f((i, j)$ - β - I - i $\text{Int}(f^{-1}(V)))$ implies that $ij \text{Int}_\delta(V) \subseteq f((i, j)$ - β - I - i $\text{Int}(f^{-1}(V)))$. Since V is ij -regular open set and hence ij - δ -open set, then $V \subseteq f((i, j)$ - β - I - i $\text{Int}(f^{-1}(V)))$ this implies that $f^{-1}(V) \subseteq (i, j)$ - β - I - i $\text{Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is an (i, j) - β - I - i -open set in X which contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence, by Theorem 4.7, f is (i, j) - β - I -almost continuous.

Theorem 4.10 For a function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

1. f is (i, j) - β - I -almost continuous.
2. (i, j) - β - I - i $\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\tau_i - \text{Cl}(V))$, for each ji - β -open set V of Y .
3. $f^{-1}(\tau_i - \text{Int}(F)) \subseteq (i, j)$ - β - I - i $\text{Int}(f^{-1}(F))$, for each ji - β -closed set F of Y .
4. $f^{-1}(\tau_i - \text{Int}(F)) \subseteq (i, j)$ - β - I - i $\text{Int}(f^{-1}(F))$, for each ji -semi closed set F of Y .
5. (i, j) - β - I - i $\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\tau_i - \text{Cl}(V))$, for each ji -semi open set V of Y .

Proof. (1) $\Rightarrow$ (2). Let V be any ji - β -open set of Y . Since $V \subseteq \tau_i - \text{Cl}(\tau_j - \text{Int}(\tau_i - \text{Cl}(V)))$ then $\tau_i - \text{Cl}(V) \subseteq \tau_i - \text{Cl}(\tau_j - \text{Int}(\tau_i - \text{Cl}(V)))$. Also $\tau_j - \text{Int}(\tau_i - \text{Cl}(V)) \subseteq \tau_i - \text{Cl}(V)$ then $\tau_i - \text{Cl}(\tau_j - \text{Int}(\tau_i - \text{Cl}(V))) \subseteq \tau_i - \text{Cl}(V)$. Then we have $\tau_i - \text{Cl}(V) = \tau_i - \text{Cl}(\tau_j - \text{Int}(\tau_i - \text{Cl}(V)))$, and hence $\tau_i - \text{Cl}(V)$ is an ij -regular closed set

in Y . Since f is (i, j) - β -I-i-almost continuous, then by Theorem 4.8, $f^{-1}(V)$ is an (i, j) - β -I-i-closed set in X . Therefore, we obtain (i, j) - β -I-i $Cl(f^{-1}(V)) \subseteq f^{-1}(\tau_i - Cl(V))$.

(2) $\Rightarrow$ (3). Let F be any ji - β -closed set of Y . Then $Y \setminus F$ is a ji - β -open set of Y and by (2), we have (i, j) - β -I-i $Cl(f^{-1}(Y \setminus F)) \subseteq f^{-1}(\tau_i - Cl(Y \setminus F)) \Leftrightarrow (i, j)$ - β -I-i $Cl(X \setminus f^{-1}(F)) \subseteq f^{-1}(Y \setminus \tau_i - Int(F)) \Leftrightarrow X \setminus (i, j)$ - β -I-i $Int(f^{-1}(F)) \subseteq X \setminus f^{-1}(\tau_i - Int(F))$. Therefore, $f^{-1}(\tau_i - Int(F)) \subseteq (i, j)$ - β -I-i $Int(f^{-1}(F))$.

(3) $\Rightarrow$ (4). This is obvious since every ji -semi closed set is ji - β -closed set.

(4) $\Rightarrow$ (5). Let V be any ji -semi open set of Y . Then $Y \setminus V$ is ji -semi closed set and by (4), we have $f^{-1}(\tau_i - Int(Y \setminus V)) \subseteq (i, j)$ - β -I-i $Int(f^{-1}(Y \setminus V)) \Leftrightarrow f^{-1}(Y \setminus \tau_i - Cl(V)) \subseteq (i, j)$ - β -I-i $Int(X \setminus f^{-1}(V)) \Leftrightarrow X \setminus f^{-1}(\tau_i - Cl(V)) \subseteq X \setminus (i, j)$ - β -I-i $Cl(f^{-1}(V))$. Therefore, (i, j) - β -I-i $Cl(f^{-1}(V)) \subseteq f^{-1}(\tau_i - Cl(V))$.

(5) $\Rightarrow$ (1). Let F be any ij -regular closed set of Y . Then F is a ji -semi open set of Y . By (5), we have (i, j) - β -I-i $Cl(f^{-1}(F)) \subseteq f^{-1}(\tau_i - Cl(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is an (i, j) - β -I-i-closed set in X . Therefore, by Theorem 4.8, f is (i, j) - β -I-i-almost continuous.

Theorem 4.11 A function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) - β -I-i-almost continuous if and only if $f^{-1}(V) \subseteq (i, j)$ - β -I-i $Int(f^{-1}(\tau_i - Int(\tau_j - Cl(V))))$ for each i -open set V of Y .

Proof. Let V be any i -open set of Y . Then $V \subseteq \tau_i - Int(\tau_j - Cl(V))$ and $\tau_i - Int(\tau_j - Cl(V))$ is ij -regular open set in Y . Since f is (i, j) - β -I-i-almost continuous, by Theorem 4.8, $f^{-1}(\tau_i - Int(\tau_j - Cl(V)))$ is an (i, j) - β -I-i-open set in X and hence we obtain that $f^{-1}(V) \subseteq f^{-1}(\tau_i - Int(\tau_j - Cl(V))) = (i, j)$ - β -I-i $Int(f^{-1}(\tau_i - Int(\tau_j - Cl(V))))$.

Conversely, Let V be any ij -regular open set of Y . Then V is i -open set of Y . By hypothesis, we have $f^{-1}(V) \subseteq (i, j)$ - β -I-i $Int(f^{-1}(\tau_i - Int(\tau_j - Cl(V)))) = (i, j)$ - β -I-i $Int(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is an (i, j) - β -I-i-open set in X and hence by Theorem 4.8, f is (i, j) - β -I-i-almost continuous.

From Theorem 4.11, the following result is obtained.

Corollary 4.12 A function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) - β -I-i-almost continuous if and only if (i, j) - β -I-i $Cl(f^{-1}(\tau_i - Cl(\tau_j - Int(F)))) \subseteq f^{-1}(F)$ for each i -closed set F of Y .

Theorem 4.13 A function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) - β -I-i-continuous if and only if f is i -continuous and for each $x \in X$ and each i -open set V of Y containing $f(x)$, there exists an (i, j) - β -I-closed set F in X containing x such that $f(F) \subseteq V$.

Proof. Let $x \in X$ and let V be any i -open set of Y containing $f(x)$. Since f is (i, j) - β -I-i-continuous, there exists an (i, j) - β -I-i-open set U of X containing x such that $f(U) \subseteq V$. Since U is (i, j) - β -I-i-open set, so for each $x \in U$, there exists an (i, j) - β -I-closed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$, and since f is (i, j) - β -I-i-continuous, then f is i -continuous.

Conversely, Let V be any i -open set of Y . We have to

show that $f^{-1}(V)$ is an (i, j) - β -I-i-open set in X . Since f is i -continuous, then $f^{-1}(V)$ is an i -open set in X . Let $x \in f^{-1}(V)$, then $f(x) \in V$. By hypothesis, there exists an (i, j) - β -I-closed set F of X containing x such that $f(F) \subseteq V$. Which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is an (i, j) - β -I-i-open set in X . Hence by Theorem 4.6, f is (i, j) - β -I-i-continuous.

Theorem 4.14 The set of all points x of X at which $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is not (i, j) - β -I-i-almost continuous is identical with the union of the (i, j) - β -I-i-boundaries of the inverse images of ij -regular open subsets of Y containing $f(x)$.

Proof. If f is not (i, j) - β -I-i-almost continuous at $x \in X$, then there exists an ij -regular open set V containing $f(x)$ such that for every (i, j) - β -I-i-open set U of X containing x , $f(U) \cap (Y \setminus V) \neq \varnothing$. This means that for every (i, j) - β -I-i-open set U of X containing x , we must have $U \cap (X \setminus f^{-1}(V)) \neq \varnothing$. Hence, it follows that $x \in (i, j)$ - β -I-i $Cl(X \setminus f^{-1}(V))$. But $x \in f^{-1}(V)$ and hence $x \in (i, j)$ - β -I-i $Cl(f^{-1}(V))$. This means that x belongs to the (i, j) - β -I-i-boundary of $f^{-1}(V)$.

Conversely, suppose that x belongs to the (i, j) - β -I-i-boundary of $f^{-1}(V_1)$ for some ij -regular open subset V_1 of Y such that $f(x) \in V_1$. Suppose that f is (i, j) - β -I-i-almost continuous at x . Then by Theorem 4.7, there exists an (i, j) - β -I-i-open set U of X containing x such that $f(U) \subseteq V_1$. Then we have $U \subseteq f^{-1}(V_1)$. This shows that $x \in (i, j)$ - β -I-i $Int(f^{-1}(V_1))$. Therefore, we have $x \notin (i, j)$ - β -I-i $Cl(X \setminus f^{-1}(V_1))$ and $x \notin (i, j)$ - β -I-i $Bd(f^{-1}(V_1))$. But this is a contradiction. This means that f is not (i, j) - β -I-i-almost continuous.

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