The Minimal System of Defining Relations of the Free Modular Lattice of Rank 3 and Lattices Close to Modular One

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Abstract We construct the system of 11 defining relations for the 3-generated free modular lattice. This system is proved to be minimal. Systems of defining relations for lattices close to modular one are studied.

Keywords Free modular lattices, Defining relations

1 Introduction and Statement of Results

Recall that the rank of a free algebra of a variety is the number of its free generators. Let A denote the free lattice of rank 3 in the variety of the modular lattices and let F be the free lattice of rank 3 in the variety of all lattices. Let f, g, h be free generators of F , and let ϕ be a homomorphism from F onto A . Then the elements $a = \phi(f)$, $b = \phi(g)$, and $c = \phi(h)$ are free generators of the lattice A . Defining relations of the lattice A in the variety of all lattices are considered in [1] and [2]. Namely, one of the results of [1] can be interpreted as the statement that the lattice A can be defined by 21 relations. In [2] it was noted that the set of these relations is not a minimal set, namely, 15 from these 21 relations were selected and the 15 they were proved to form a minimal set of the defining relations of A . Note that in [1] it was presented the set of 7 defining relations of the free distributive lattice of rank 3. In [3] this set is proved to be minimal.

The main result of the present paper is that we have found a set of 11 defining relations for the lattice A . Note that this set is not a subset of the defining relations from [1].

We list these relations:

$$\begin{aligned} (a \vee b) \wedge (a \vee c) \wedge (b \vee c) = \\ = ((a \wedge (b \vee c)) \vee (c \wedge (a \vee b))) \wedge \\ \wedge ((b \wedge (a \vee c)) \vee (c \wedge (b \vee a))) \wedge \\ \wedge ((a \wedge (c \vee b)) \vee (b \wedge (a \vee c))) \end{aligned} \quad (1)$$

$$\begin{aligned} (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = \\ = ((a \vee (b \wedge c)) \wedge (c \vee (a \wedge b))) \vee \\ \vee ((b \vee (a \wedge c)) \wedge (c \vee (b \wedge a))) \vee \\ \vee ((a \vee (c \wedge b)) \wedge (b \vee (a \wedge c))) \end{aligned} \quad (2)$$

$$(a \vee (b \wedge c)) \wedge (b \vee c) = (a \wedge (b \vee c)) \vee (b \wedge c) \quad (3)$$

$$(b \vee (c \wedge a)) \wedge (c \vee a) = (b \wedge (c \vee a)) \vee (c \wedge a) \quad (4)$$

$$(c \vee (a \wedge b)) \wedge (a \vee b) = (c \wedge (a \vee b)) \vee (a \wedge b) \quad (5)$$

$$(a \vee b) \wedge (a \vee c) = a \vee ((a \vee b) \wedge (a \vee c) \wedge (b \vee c)) \quad (6)$$

$$(b \vee a) \wedge (b \vee c) = b \vee ((a \vee b) \wedge (a \vee c) \wedge (b \vee c)) \quad (7)$$

$$(c \vee a) \wedge (c \vee b) = c \vee ((a \vee b) \wedge (a \vee c) \wedge (b \vee c)) \quad (8)$$

$$(a \wedge b) \vee (a \wedge c) = a \wedge ((a \wedge b) \vee (a \wedge c) \vee (b \wedge c)) \quad (9)$$

$$(b \wedge a) \vee (b \wedge c) = b \wedge ((a \wedge b) \vee (a \wedge c) \vee (b \wedge c)) \quad (10)$$

$$(c \wedge a) \vee (c \wedge b) = c \wedge ((a \wedge b) \vee (a \wedge c) \vee (b \wedge c)) \quad (11)$$

Denote the set of these relations by ρ .

Theorem 1. The lattice generated by the elements a, b, c subject to the defining relations ρ is isomorphic to the free modular lattice of rank 3. If a set σ of relations is strictly contained in ρ , then the lattice generated by a, b, c subject to the defining relations σ is not modular.

In light of this theorem there is natural interest in lattices generated by a, b, c subject to one or another subset σ of the defining relations from the set ρ . In particular, how much they differ from the free modular lattice of rank 3. In the following theorem the answer is given for σ consisting of relations (1) and (2).

Theorem 2. The lattice generated by the elements a, b, c subject to the defining relations (1) and (2) is isomorphic to the lattice represented on Figure 1.

In reality, Theorem 2 makes it easy to get a description of the lattice generated by the elements a, b and c subject to a set σ of relations when $\{(1), (2)\} \subset \sigma \subset \rho$.

Theorem 1 was announced in [4] without proof.

Figure 1. The lattice which is close to modular one

2 Proofs

Let a, b, c be any elements of a lattice. Further we use the following notation:

$$a_1 = (a \vee (b \wedge c)) \wedge (b \vee c); \quad (12)$$

$$a_2 = (a \wedge (b \vee c)) \vee (b \wedge c); \quad (13)$$

$$b_1 = (b \vee (c \wedge a)) \wedge (c \vee a); \quad (14)$$

$$b_2 = (b \wedge (c \vee a)) \vee (c \wedge a); \quad (15)$$

$$c_1 = (c \vee (a \wedge b)) \wedge (a \vee b); \quad (16)$$

$$c_2 = (c \wedge (a \vee b)) \vee (a \wedge b); \quad (17)$$

$$t = (a \vee b) \wedge (a \vee c) \wedge (b \vee c); \quad (18)$$

$$s = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c). \quad (19)$$

Lemma 1. The following inequalities hold in any lattice:

$$s \leq a_2 \leq a_1 \leq t;$$

$$s \leq b_2 \leq b_1 \leq t;$$

$$s \leq c_2 \leq c_1 \leq t.$$

Proof. The distributive inequality [5, Lemma I.4.9] shows that

$$(a \vee (b \wedge c)) \wedge (b \vee c) \leq (a \vee b) \wedge (a \vee c) \wedge (b \vee c),$$

i.e. $a_1 \leq t$. Passing on to the dual lattice we get $s \leq a_2$. At the same time, since $b \wedge c \leq b \vee c$ the same inequality shows that

$$(a \wedge (b \vee c)) \vee (b \wedge c) \leq (a \wedge (b \wedge c)) \wedge (b \vee c),$$

i.e. $a_2 \leq a_1$. Consequently, the inequalities $s \leq a_2 \leq a_1 \leq t$ hold in any lattice.

Other inequalities can be derived by cyclic substitutions of elements a, b and c .

Lemma 2. Let L be a lattice generated by a, b, c . If relation (2) holds for elements a, b, c then

$$(a \vee (b \wedge c)) \wedge (c \vee (a \wedge b)) = s;$$

$$(a \vee (b \wedge c)) \wedge (b \vee (a \wedge c)) = s;$$

$$(b \vee (a \wedge c)) \wedge (c \vee (a \wedge b)) = s.$$

Proof. It is clear that $a \vee (b \wedge c) \geq s$ and $c \vee (a \wedge b) \geq s$, therefore $(a \vee (b \wedge c)) \wedge (c \vee (a \wedge b)) \geq s$. But from (2) it follows that $(a \vee (b \wedge c)) \wedge (c \vee (a \wedge b)) \leq s$, hence the first equality holds. Other equalities are derived by cyclic substitutions of elements a, b and c .

From (1) the statement dual of Lemma 2 is proved similarly.

Let L denote the lattice generated by a, b, c . And the defining relations (1) and (2) holds for elements a, b, c in L . Let us consider the following subsets of the lattice L :

$$M_1 = \{a \vee b \vee c, a \vee c, a \vee b, b \vee c, (a \vee b) \wedge (a \vee c),$$

$$(a \vee c) \wedge (b \vee c), (a \vee b) \wedge (b \vee c), a \vee t, c \vee t, b \vee t, t\};$$

$$M_2 = \{a \wedge b \wedge c, a \wedge c, a \wedge b, b \wedge c, (a \wedge b) \vee (a \wedge c),$$

$$(a \wedge c) \vee (b \wedge c), (a \wedge b) \vee (b \wedge c), a \wedge s, b \wedge s, c \wedge s, s\};$$

$$M_a = \{a \vee (b \wedge c), a, a_1, a_2, a \wedge (b \vee c)\};$$

$$M_b = \{b \vee (a \wedge c), b, b_1, b_2, b \wedge (a \vee c)\};$$

$$M_c = \{c \vee (b \wedge a), c, c_1, c_2, c \wedge (b \vee a)\}.$$

Lemma 3. The set $M_1 \cup M_2 \cup M_a \cup M_b$ is a sublattice of the lattice L .

Proof. Since the set $M_1 \cup M_2 \cup M_a \cup M_b$ is self-dual and relations (1) and (2) are dual to one another it is sufficient to check on the closure of this set under the operation $\wedge$. Also it is clear that we have to consider those pairs of elements that are not known to be comparable. The check will be performed in several steps.

The set $M_1 \cup M_2 \cup M_a \cup M_b$ is shown on the Figure 2. Here we don't denote the elements $(b \vee c) \wedge (a \vee c)$ and $(b \wedge c) \vee (a \wedge c)$ to do not overload the figure. If $x \geq y$ for some x and y from the lattice L they are joined by a line and the element x is above than y . The inequalities between a_1 and a_2 , b_1 and b_2 , c_1 and c_2 , t and s have been proved in Lemma 1. Other inequalities are well known or checked easy. But we do not guarantee as yet that all elements shown in this diagram are different.

Let us carry out the necessary calculations.

1) $(a \vee c) \wedge ((a \vee b) \wedge (b \vee c)) = t$. It means that $x \wedge y = t$ for any element x from the interval $[t, a \vee c]$ and for any element y from the interval $[t, (a \vee b) \wedge (b \vee c)]$.

2) $(a \vee (b \wedge c)) \wedge (b \vee c) = a_1$. Then $(a \vee (b \wedge c)) \wedge x = a_1$ for any element x from the interval $[a_1, b \vee c]$.

3) $a \wedge x = a \wedge (b \vee c)$ for any element x from the interval $[a \wedge (b \vee c), b \vee c]$.

4) $(a \vee (b \wedge c)) \wedge (b \vee (a \wedge c)) = s$ by Lemma 2. It means that the set $\{a \vee (b \wedge c), a_1, a_2, s, b_2, b_1, b \vee (a \wedge c)\}$ is closed under the operation $\wedge$.

5) The equality $(a \vee (b \wedge c)) \wedge (b \vee (a \wedge c)) = s$ implies $(a \vee (b \wedge c)) \wedge b = s \wedge b$. Then the set $\{a \vee (b \wedge c), a_1, a_2, s, b \vee s, b \wedge (a \vee c), b\}$ is closed under the operation $\wedge$.

6) $x \wedge y = a \wedge b$ for any element x from the interval $[a \wedge b, a]$ and for any element y from the interval $[a \wedge b, b]$.

7) $a \wedge (c \wedge s) = a \wedge c$. It means the closure of the set $\{a, a \wedge (b \wedge c), a \wedge s, (a \wedge b) \vee (a \wedge c), a \wedge c, (a \wedge c) \vee (b \wedge c), c \wedge s\}$ under the operation $\wedge$.

8) $(b \wedge c) \wedge x = a \wedge b \wedge c$ for any element x from the interval $[a \wedge b \wedge c, a]$.

The roles of a and b in $M_1 \cup M_2 \cup M_a \cup M_b$ are interchangeable, and so are the roles of a, b and c in $M_1 \cup M_2$. Hence the verifications made above suffice to claim the closeness of the set $M_1 \cup M_2 \cup M_a \cup M_b$ under the operation $\wedge$.

In the proof of Lemma 4 we use the following general-algebraic observation. Let U be an universal algebra

with binary operation $*$ and let $\{K_i \mid i \in I\}$ be a family of subsets of the algebra U . Suppose that for any pair of subsets K_i and K_j (which are not necessarily different) from this family, their union $K_i \cup K_j$ is closed under this operation. Then $\cup\{K_i \mid i \in I\}$ is also closed under the operation $*$ too. Indeed, if $x, y \in \cup\{K_i \mid i \in I\}$ then there exist i and j that $x \in K_i$ and $y \in K_j$. Hence $x * y \in K_i \cup K_j \subseteq \cup\{K_i \mid i \in I\}$.

Figure 2. The set $M_1 \cup M_2 \cup M_a \cup M_b$

Lemma 4. The set $M_1 \cup M_2 \cup M_a \cup M_b \cup M_c$ is a sublattice of the lattice L .

Proof. Let $P = M_1 \cup M_2 \cup M_a$, $Q = M_1 \cup M_2 \cup M_b$ and $R = M_1 \cup M_2 \cup M_c$. By Lemma 3 the sets $P \cup Q$, $P \cup R$, $Q \cup R$ are closed under $\wedge$ and $\vee$. Consequently the set $P \cup Q \cup R$ is closed under these operations too, i.e. this set is a sublattice of the lattice L .

Below we will use the following remark well known from general algebra. Let L_1 and L_2 be lattices with the same set of generating elements and let τ_1 and τ_2 be sets of defining relations, respectively, for these lattices. In this case if $\tau_1 \subseteq \tau_2$ then the identity mapping of generating elements of lattice L_1 on generating elements of lattice L_2 can be continued to a surjective homomorphism of L_1 on L_2 .

Before proving Theorem 2 let us consider the lattice which is mentioned in this theorem. For convenience we denote it by L_1 . It has 37 elements. Also it is obvious that this lattice is generated by three elements. On Figure 3 we have denoted them as a , b and c . Other

elements of the lattice have been evaluated by this generators and they have been marked by corresponding formulas apart from elements $c \vee (a \wedge b)$, $c \wedge (a \vee b)$, $(a \vee c) \wedge (b \vee c)$ and $(a \wedge c) \vee (b \wedge c)$. As above a_1 , a_2 , b_1 , b_2 , c_1 , c_2 , t and s denote the elements with respect to formulas (12) — (19).

Of course, for some elements of L_1 we can write also other variants of their representations by the generating elements a , b and c . For example, $t = a_2 \vee b_2 = a_2 \vee c_2 = b_2 \vee c_2$. Each of these equations is a certain relation between the generating elements a , b , c of the lattice L_1 .

Figure 3. The set $M_1 \cup M_2 \cup M_a \cup M_b \cup M_c$

Let τ denote the set of all relations of the generated elements a , b and c of the lattice L_1 .

Proof of Theorem 2. Let L_2 be a lattice generated by elements a , b and c with the defining relations (1) and (2). The lattice L_1 is the homomorphic image of the lattice L_2 since relations (1) and (2) are contained in τ . Consequently there are at least 37 different elements in the lattice L_2 .

In the same time by Lemma 4 the set

$$M_1 \cup M_2 \cup M_a \cup M_b \cup M_c$$

is a sublattice of the lattice L_2 . This sublattice contains the elements a , b and c , hence L_2 contains at most 37 elements. It means that the lattices L_1 and L_2 are isomorphic.

Proof of Theorem 1. Let L_3 be the lattice generated by elements a, b and c with the defining relations ρ . Let ϕ be the identity mapping from the generating elements a, b, c of L_2 onto the generating elements a, b, c of L_3 . Since the relations (1) and (2) are contained in ρ , the mapping ϕ can be continued to a surjective homomorphism from L_2 onto L_3 . This homomorphism identifies elements at least in 9 pairs: a_1 and a_2 , b_1 and b_2 , c_1 and c_2 because (3) — (5) are true, $a \vee t$ and $(a \vee b) \wedge (a \vee c)$, $b \vee t$ and $(b \vee a) \wedge (b \vee c)$, $c \vee t$ and $(c \vee a) \wedge (c \vee b)$ because (6) — (8) are true, and also three pairs of dual elements because (9) — (11) are true. This means that the number of elements of L_3 is at most 28.

Also all relations from ρ are true for any modular lattice (e.g. [5]). Therefore there exists surjective homomorphism φ from lattice L_3 onto lattice A that contains exactly 28 elements. Consequently φ is an isomorphism.

For the proof of the second statement, for any set σ of defining relations obtained from ρ by removing one of its elements we construct a non-modular lattice satisfying the relations in σ . For example, the lattice in Figure 4 satisfies the relations (2) — (11) but it does not satisfy the relation (1).

Figure 4. The lattice which does not satisfy the relation (1)

The lattice in Figure 5 is not modular and satisfies the relations (1), (2), and (4) — (11) but it does not satisfy the relation (3).

The lattice in Figure 6 satisfies the relations (1) — (7) and (9) — (11) but it does not satisfy the relation (8). The element $(b \vee c) \wedge (a \vee c)$ is not denoted to do not overload the figure.

All other examples can be obtained from the ones presented in Figures 4 — 6 by a suitable renaming of elements and passing, if necessary, to dual lattices. To show independence of the relation (3) from the other 10 relations it is possible to give an example of lattice with a smaller number of elements (for example, such as in [4]). But the lattice of Figure 5 is the lattice generated by the elements a, b, c subject to the defining relations (1), (2), (4) — (11). That is why it is interesting. The proof is similar to the proof of Theorem 1 using the result of Theorem 2. In this regard Figure 6 gives the lattice generated by the elements a, b, c subject to the defining relations (1) — (7), (9) — (11).

Using Theorem 2 it is easy, in the same way, to obtain the lattice generated by the elements a, b, c subject to

Figure 5. The lattice which does not satisfy the relation (3)

Figure 6. The lattice which does not satisfy the relation (8)

the set of defining relations σ where $\{(1), (2)\} \subset \sigma \subset \rho$.

3 Discussion

We note that in connection with this paper it would be interesting to find out whether there exists a set of defining relations for the free modular lattices of rank 3 in which the number of relations will be less than 11.

Although the defining relations (1) and (2) gives the nonmodular lattice it is finite. Whether the lattice generated by the elements a, b, c subject to the defining relations (2) — (11) finite? It is natural to expect a negative response but it is unknown to us.

Lemma 2 shows that the relation (2) considered as the identity is equivalent to the identity $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (y \vee (x \wedge z)) \wedge (z \vee (x \wedge y))$. In turn this identity is equivalent to the modularity of a lattice. Indeed if $x \leq y$ then $(y \vee (x \wedge z)) \wedge (z \vee (x \wedge y)) = y \wedge (z \vee x)$ and $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = x \vee (y \wedge z)$. It means that the modularity follows from the identity. Conversely if a lattice is modular then the equalities $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (y \wedge (z \vee (x \wedge y))) \vee (x \wedge z) = (y \vee (x \wedge z)) \wedge (z \vee (x \wedge y))$ follow from inequalities $y \wedge y \leq x$ and $y \wedge z \leq z \leq z \vee (x \wedge y)$. In the same time the variety $\mathcal{V}$ generated by the free modular lattice of rank 3 is a subvariety of all modular lattices. This variety is clear to be contained in the variety $\mathcal{W}$ generated by the lattice on Figure 1. Since the lattice on Figure 1 is close to the free modular lattice of rank 3 we can expect that in the lattice of varieties the interval $[\mathcal{V}, \mathcal{W}]$ is not very long.

Acknowledgements

We would like to express our thanks to Lev Shevrin for the discussions that contributed to significant improvements of the text.

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