On Some New Inequalities for Differentiable \((h_1, h_2)\) – Preinvex Functions on the Co-Ordinates

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Abstract We consider and study a new class of convex functions that are called \((h_1, h_2)\) - preinvex functions on the co-ordinates. Some Hermite-Hadamard inequalities for the \((h_1, h_2)\) – preinvex functions on the co-ordinates and its variant forms are derived. Some our theorems are new and other generalize some results of Dragomir and Latif.

Keywords \((h_1, h_2)\) - Preinvex Function on the Co-Ordinates, Hermite-Hadamard Type Inequality, Convex Function

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1. Introduction

Let \(f: I \subseteq \mathbb{R} \to \mathbb{R}\) be a convex function defined on the interval \(I\) of real numbers and \(a < b\). The following double inequality:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}
\]

is well known in the literature as Hermite-Hadamard inequalities.

Similar inequalities were obtained for \(s\)-convex function by Dragomir and Fitzpatrick in [1]; for \(h\)-convex function by Sarıkaya, Saglam and Yıldırım in [2]; for preinvex function by Noor in [3]; for \(h\)-preinvex function by Matłoka in [4]; for preinvex function integrals by Iscan in [5].

A modification for convex functions, which is also known as co-ordinated convex functions, was introduced by Dragomir [6]. In the same article, Dragomir established the following Hermite-Hadamard type inequalities for convex functions on the co-ordinates:

\[
f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y)dxdy
\]

is well known in the literature as Hermite-Hadamard inequalities.

\[
\leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4}.
\]

In [7] Alomari and Darus proved the following inequality for an \(s\)-convex function on the co-ordinates:

\[
4^{s-1}f\left(\frac{a + b + c + d}{2}\right) \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y)dxdy
\]

\[
\leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{s + 1}.
\]

In the paper [4] Matłoka proved that for \((h_1, h_2)\) - preinvex function on the co-ordinates the following inequality holds:

\[
\frac{1}{4h_1(\frac{1}{2}h_2)}f\left(a + \frac{1}{2}\eta_1(b, a), c + \frac{1}{2}\eta_2(d, c)\right)
\]

\[
\leq \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^b \int_c^d f(x, y)dxdy
\]

\[
\leq \left[ f(a, c) + f(b, c) + f(a, d) + f(b, d) \right]
\]

\[
\cdot \int_0^1 h_1(t_1)dt_1 \int_0^1 h_2(t_2)dt_2.
\]

For the formal definition of \((h_1, h_2)\) - preinvex function on the co-ordinates see in the next part of this paper.

Some interesting inequalities for co-ordinated convex functions were proved by Sarıkaya, Set, Özdemir and Dragomir in [8].

The main purpose of this paper is to establish new inequalities for differentiable \((h_1, h_2)\) - preinvex functions on the co-ordinates. Some theorems are new and other generalize Theorems 2, 3, 4 obtained by Latif and Dragomir in [9]. Throughout this paper, we assume that considered integrals exist.

2. Main Results

Let \(f: X \to \mathbb{R}\) and \(\eta: X \times X \to \mathbb{R}^n\), where \(X\) is a nonempty set in \(\mathbb{R}^n\), be continuous functions. First, we recall
the following well-known results and concepts; see [4, 10, 11, 12] and the references therein.

Definition 2.1 Let \( u \in X \). Then the set \( X \) is said to be invex at \( u \) with respect to \( \eta \), if

\[
u + t\eta(v, u) \in X,\]

for all \( v \in X \) and \( t \in [0,1] \).

\( X \) is said to be an invex set with respect to \( \eta \), if \( X \) is invex at each \( u \in X \).

Definition 2.2 The function \( f \) on the invex set \( X \) is said to be preinvex with respect to \( \eta \), if

\[
f(u + t\eta(v, u)) \leq (1 - t)f(u) + tf(v)
\]

for all \( u, v \in X \) and \( t \in [0,1] \).

Definition 2.3 Let \( h: [0,1] \rightarrow \mathbb{R} \) be a non-negative function, \( h \neq 0 \). The non-negative function \( f \) on the invex set \( X \) is said to be \( h \)-preinvex with respect to \( \eta \), if

\[
f(u + t\eta(v, u)) \leq h(1 - t)f(u) + h(t)f(v),
\]

for each \( u, v \in X \) and \( t \in [0,1] \).

Let us note that:

- if \( \eta(v, u) = v - u \) then we get the definition of \( h \)-convex function introduced by Varošanec in [13];
- if \( h(t) = t \) then our definition reduces to the definition of preinvex function;
- if \( \eta(v, u) = v - u \) and \( h(t) = t \), then we obtain the definition of convex function.

Now let \( X_1 \) and \( X_2 \) be nonempty subsets of \( \mathbb{R}^n \), let \( \eta_1: X_1 \times X_1 \rightarrow \mathbb{R}^n \) and \( \eta_2: X_2 \times X_2 \rightarrow \mathbb{R}^n \).

Definition 2.4[4] Let \((u, v) \in X_1 \times X_2 \). We say \( X_1 \times X_2 \) is invex at \((u, v)\) with respect to \( \eta_1 \) and \( \eta_2 \) if for each \((x, y) \in X_1 \times X_2 \) and \( t_1, t_2 \in [0, 1] \),

\[
(u + t_1\eta_1(x, u) + t_2\eta_2(y, v)) \in X_1 \times X_2.
\]

\( X_1 \times X_2 \) is said to be an invex set with respect to \( \eta_1 \) and \( \eta_2 \) if \( X_1 \times X_2 \) is invex at each \((u, v) \in X_1 \times X_2 \).

Definition 2.5[4] Let \( h_1 \) and \( h_2 \) be the non-negative functions on \([0,1] \), \( h_1 \neq 0 \), \( h_2 \neq 0 \). The non-negative function \( f \) on the invex set \( X_1 \times X_2 \) is said to be co-ordinated \((h_1, h_2)\) - preinvex with respect to \( \eta_1 \) and \( \eta_2 \) if the partial mappings \( f_1: X_1 \rightarrow \mathbb{R} \), \( f_2(x) = f(x, y) \) and \( f_2: X_2 \rightarrow \mathbb{R} \), \( f_1(y) = f(x, y) \) are \( h_1 \)-preinvex with respect to \( \eta_1 \) and \( h_2 \)-preinvex with respect to \( \eta_2 \) respectively, for all \( y \in X_2 \) and \( x \in X_1 \).

Remark 1. From the above definition it follows that if \( f \) is co-ordinated \((h_1, h_2)\) - preinvex function then

\[
f(x + t_1h_1(b, x), y + t_2h_2(d, y)) \leq h_1(1 - t_1)f(x, y + t_2h_2(d, y)) + h_1(t_1)f(b, y + t_2h_2(d, y))
\]

\[
\leq h_1(1 - t_1)h_2(1 - t_2)f(x, y) + h_1(1 - t_1)h_2(2t_2)f(x, d)
\]

... and \( h_1(t_1)h_2(1 - t_2)f(b, y) + h_1(t_1)h_2(2t_2)f(b, d) \).

Remark 2. Let us note that if \( \eta_1(x, u) = x - u \), \( \eta_2(y, v) = y - v \), \( t_1 = t_2 \) and \( h_1(t) = h_2(t) = t \), then our definition of co-ordinated \((h_1, h_2)\) - preinvex function reduces to the definition of convex function on the co-ordinates proposed by Dragomir [1]. Moreover, if \( h_1(t) = h_2(t) = t^p \), then our definition reduces to the definition of \( s \)-convex function on the co-ordinates proposed by Alomari and Darus [7].

Let us consider a bidimensional interval \( \Delta = [a, a + \eta_1(b, a) \times c, c + \eta_2(d, c)] \in \mathbb{R}^2 \) with \( a < a^* + \eta_1(b, a) \) and \( c < c + \eta_2(d, c) \).

Lemma 1. Let \( f: \Delta \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta \). If \( \frac{\partial f}{\partial t_2} \in L(\Delta) \), then the following equality holds:

\[
f(a, c) + \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_0^{\eta_1(b, a)} \int_a^{\eta_2(d, c)} f(x, y)\,dx\,dy - \frac{1}{\eta_1(b, a)} \int_a^{\eta_1(b, a)} f(x, c)\,dx
\]

\[
- \frac{1}{\eta_2(d, c)} \int_c^{\eta_2(d, c)} f(a, y)\,dy
\]

\[
= \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_0^1 \int_0^1 (1 - t_1) (1 - t_2) \frac{\partial^2 f}{\partial t_2 \partial t_1} (a + t_1\eta_1(b, a), c + t_2\eta_2(d, c))\,dt_1\,dt_2.
\]

Proof. By integration by parts, we get

\[
\int_0^1 \int_0^1 (1 - t_1) (1 - t_2) \frac{\partial^2 f}{\partial t_2 \partial t_1} (a + t_1\eta_1(b, a), c + t_2\eta_2(d, c))\,dt_1\,dt_2
\]

\[
= \int_0^1 (1 - t_2) \left[ \left. \frac{\partial f}{\partial t_1} (a + t_1\eta_1(b, a), c + t_2\eta_2(d, c)) \right|_0^1 \right. + t_2\eta_2(d, c) \right. + \frac{1}{\eta_1(b, a)} \int_0^1 \frac{\partial f}{\partial t_2} (a + t_1\eta_1(b, a), c + t_2\eta_2(d, c))\,dt_1\,dt_2
\]

\[
= \int_0^1 (1 - t_2) \left[ \left. \frac{\partial f}{\partial t_1} (a, c + t_2\eta_2(d, c)) \right|_0^1 \right. + \frac{1}{\eta_1(b, a)} \int_0^1 \frac{\partial f}{\partial t_2} (a + t_1\eta_1(b, a), c + t_2\eta_2(d, c))\,dt_1\,dt_2
\]

Thus, again by integration by parts in the right hand side of (2), it follows that
\[
\int_0^1 (1 - t_2) \left( 1 - \theta \frac{\partial f}{\partial t_2}(a + t_1 \eta_1(b, a), c + t_2 \eta_2(d, c)) \right) dt_2
\]

Proof. From Lemma 1, we have

\[
\int f(a, c) + \frac{1}{\eta_1(b, a) \eta_2(d, c)} \int_a^c f(x, y) dxy - A \leq \eta_1(b, a) \eta_2(d, c) \int_0^1 (1 - t_1)(1 - t_2) \]

Since \( \frac{\partial^2 f}{\partial t_2 \partial t_1} \) is an \((h_1, h_2)\) - preinvex function on the co-ordinates, then one has:

\[
\int f(a, c) + \frac{1}{\eta_1(b, a) \eta_2(d, c)} \int_a^c f(x, y) dxy - A \leq \eta_1(b, a) \eta_2(d, c) \int_0^1 (1 - t_2) \]

\[
\times \int_0^1 (1 - t_1) \left( h_1(t_1) \int_0^1 (1 - t_1) \left( \frac{\partial^2 f}{\partial t_2 \partial t_1}(a + t_2 \eta_2(d, c)) \right) dt_2 \right) dt_1
\]

\[
= \eta_1(b, a) \eta_2(d, c) \int_0^1 (1 - t_2) \left( \frac{\partial^2 f}{\partial t_2 \partial t_1}(a, c + t_2 \eta_2(d, c)) \right) dt_2
\]

Theorem 1. Let \( f: \Delta \to R \) be a partial differentiable mapping on \( \Delta \). If \( \frac{\partial^2 f}{\partial t_2 \partial t_1} \) is an \((h_1, h_2)\) - preinvex function on the co-ordinates on \( \Delta \), then one has the inequalities

\[
\int f(a, c) + \frac{1}{\eta_1(b, a) \eta_2(d, c)} \int_a^c f(x, y) dxy - A \leq \eta_1(b, a) \eta_2(d, c) \int_0^1 (1 - t_2)
\]

\[
\times \int_0^1 (1 - t_1) \left( h_1(t_1) \int_0^1 (1 - t_1) \left( \frac{\partial^2 f}{\partial t_2 \partial t_1}(a, c + t_2 \eta_2(d, c)) \right) dt_2 \right) dt_1
\]

\[
= \eta_1(b, a) \eta_2(d, c) \int_0^1 (1 - t_2) \left( \frac{\partial^2 f}{\partial t_2 \partial t_1}(a, c + t_2 \eta_2(d, c)) \right) dt_2
\]
\[
+ \int_0^1 (1-t_1) h_1(t_1) \, dt_1 \int_0^1 (1-t_2) \, dt_2 \\
+ h_2(t_2) \left( \frac{\partial^2 f}{\partial t_2 \partial t_1^2} (a, d) \right) dt_2 + \int_0^1 (1-t_1) h_1(t_1) \, dt_1 \\
+ \frac{\partial^2 f}{\partial t_2 \partial t_1^2} (b, c) \int_0^1 (1-t_2) h_2(t_2) (1-t_2) \, dt_2 \\
+ \int_0^1 (1-t_1) h_1(t_1) \, dt_1 \\
+ \frac{\partial^2 f}{\partial t_2 \partial t_1^2} (b, d) \int_0^1 (1-t_1) h_1(t_1) \, dt_1 \\
+ \frac{\partial^2 f}{\partial t_2 \partial t_1^2} (a, c) \int_0^1 (1-t_2) h_2(t_2) \, dt_2 \\
= \eta_1(b, a) \eta_2(d, c) \left( \frac{\partial^2 f}{\partial t_2 \partial t_1^2} (a, c) \right) \int_0^1 t_1 h_1(t_1) \, dt_1 \int_0^1 t_2 h_2(t_2) \, dt_2 \\
+ \frac{\partial^2 f}{\partial t_2 \partial t_1^2} (a, d) \int_0^1 t_1 h_1(t_1) \, dt_1 \int_0^1 (1-t_2) h_2(t_2) \, dt_2 \\
+ \frac{\partial^2 f}{\partial t_2 \partial t_1^2} (b, d) \int_0^1 (1-t_1) h_1(t_1) \, dt_1 \\
+ \frac{\partial^2 f}{\partial t_2 \partial t_1^2} (b, c) \int_0^1 (1-t_1) h_1(t_1) \, dt_1 \int_0^1 t_2 h_2(t_2) \, dt_2 \frac{1}{1-q} \\
\]
+\eta_1(t_1)h_2(t_2) \left| \frac{\partial^2 f}{\partial t_2 \partial t_1} (b, c) \right|^q \\
+ \frac{\partial^2 f}{\partial t_2 \partial t_1} (b, d) \right|^q \\
+ h_1(t_1)h_2(t_2) \left| \frac{\partial^2 f}{\partial t_2 \partial t_1} (b, c) \right|^q \\
+ h_1(t_1)h_2(t_2) \left| \frac{\partial^2 f}{\partial t_2 \partial t_1} (b, d) \right|^q \\
\times \int_0^1 (1 - t_2)h_2(t_2) dt_2 \right) \right)^{\frac{1}{\eta}}.

In the analogous way by using the well-known Hölder inequality for double integrals we can prove the following theorem.

**Theorem 3.** Let \( f: \Delta \to R \) be a partial differentiable mapping on \( \Delta \). If \( \frac{\partial^2 f}{\partial t_2 \partial t_1}^q > 1 \), is an \( (h_1, h_2) \) -preinvex function on the co-ordinates on \( \Delta \), then one has:

\[
\frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} f(x, y) dx dy - A \
\]

\[
\leq \frac{1}{(p + 1)^2} \left( \frac{\partial^2 f}{\partial t_2 \partial t_1} (a, c) \right) \int_0^1 t_1 h_1(t_1) dt_1 \int_0^1 (1 - t_2)h_2(t_2) dt_2 \\
+ \frac{\partial^2 f}{\partial t_2 \partial t_1} (b, c) \int_0^1 (1 - t_1)h_1(t_1) dt_1 \int_0^1 t_2h_2(t_2) dt_2 \\
+ \frac{\partial^2 f}{\partial t_2 \partial t_1} (b, d) \int_0^1 (1 - t_1)h_1(t_1) dt_1 \int_0^1 (1 - t_2)h_2(t_2) dt_2 \\
\times \int_0^1 (1 - t_2)h_2(t_2) dt_2 \right) \right)^{\frac{1}{\eta}}.
\]

where

\[
A = \frac{1}{\eta_1(b, a) \eta_2(d, c)} \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} f(x, c) dx \\
+ \frac{1}{\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(a, y) dy \\
\]

and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Lemma 2.** Let \( f: \Delta \to R \) be a partial differentiable mapping on \( \Delta \). If \( \frac{\partial^2 f}{\partial t_2 \partial t_1} \in L(\Delta) \), then the following equality holds:

\[
f \left( a + \frac{1}{2} \eta_1(b, a), c + \frac{1}{2} \eta_2(d, c) \right) \\
- \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f \left( x, c + \frac{1}{2} \eta_2(d, c) \right) dx \\
\]
\[- \frac{1}{\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f\left(a + \frac{1}{2} \eta_1(b, a), y\right) dy + \frac{1}{\eta_1(b, a) \eta_2(d, c)} \int_a^{a+\eta_1(b, a) c + \eta_2(d, c)} f(x, y) dx dy = \eta_1(b, a) \eta_2(d, c) \int_0^1 p(t_1) q(t_2) \frac{\partial f}{\partial t_2} (a + t_1 \eta_1(b, a), c + t_2 \eta_2(d, c)) dt_1 dt_2 \]

where

\[ p(t_1) = \begin{cases} t_1, & t_1 \in [0, \frac{1}{2}] \\ t_1 - 1, & t_1 \in [\frac{1}{2}, 1] \end{cases} \]

and

\[ q(t_2) = \begin{cases} t_2, & t_2 \in [0, \frac{1}{2}] \\ t_2 - 1, & t_2 \in [\frac{1}{2}, 1] \end{cases} \]

Proof. Integration by parts, we can write

\[- \frac{1}{\eta_1(b, a) \eta_2(d, c)} \int_0^1 f(t_1) \frac{\partial f}{\partial t_1} (a + t_1 \eta_1(b, a), c + t_2 \eta_2(d, c)) dt_1 dt_2 + (t_1 - 1) \int_0^1 f(a + t_1 \eta_1(b, a), c + t_2 \eta_2(d, c)) dt_1 \frac{\partial f}{\partial t_2} (a + t_1 \eta_1(b, a), c + t_2 \eta_2(d, c)) dt_2 \]

= \frac{1}{\eta_1(b, a) \eta_2(d, c)} \left\{ \int_0^1 f(a + \frac{1}{2} \eta_1(b, a), c + t_2 \eta_2(d, c)) dt_2 \right\} \frac{\partial f}{\partial t_2} (a + \frac{1}{2} \eta_1(b, a), c + t_2 \eta_2(d, c)) dt_2 + (t_1 - 1) \int_0^1 f(a + \frac{1}{2} \eta_1(b, a), c + t_2 \eta_2(d, c)) dt_1 \frac{\partial f}{\partial t_2} (a + \frac{1}{2} \eta_1(b, a), c + t_2 \eta_2(d, c)) dt_2 \]
\[
\frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^c f \left( a + \frac{1}{2} \eta_1(b,a), c + \frac{1}{2} \eta_2(d,c) \right) \, dt_1 \nabla \frac{1}{\eta_1(b,a)} \int_a^c f \left( a + t_1 \eta_1(b,a), c + \frac{1}{2} \eta_2(d,c) \right) \, dt_1 \\
+ \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^c f(x,y) \, dy
\]

Using the change of the variable \( x = a + t_1 \eta_1(b,a) \) and \( y = c + t_2 \eta_2(d,c) \), then multiplying both sides with \( \eta_1(b,a) \cdot \eta_2(d,c) \), this completes the proof.

**Theorem 4.** Let \( f: \Delta \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta \). If \( \frac{\partial^2 f}{\partial t_2 \partial t_1} \) is an \( (h_1, h_2) \) - preinvex function on the co-ordinates on \( \Delta \), then the following inequality holds:

\[
\frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^c f \left( a + \frac{1}{2} \eta_1(b,a), c + \frac{1}{2} \eta_2(d,c) \right) \, dx \\
- \frac{1}{\eta_1(b,a)} \int_a^c f \left( x, c + \frac{1}{2} \eta_2(d,c) \right) \, dx \\
+ \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^c f(x,y) \, dy
\]

By computing the integrals, we obtain that all of the above double integrals are equal to

\[
\frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^c f \left( a + \frac{1}{2} \eta_1(b,a), c + \frac{1}{2} \eta_2(d,c) \right) \, dy
\]

which completes the proof.

**Theorem 5.** Let \( f: \Delta \to \mathbb{R} \) be a partial differentiable on \( \Delta \). If \( \frac{\partial^2 f}{\partial t_2 \partial t_1} \) is an \( (h_1, h_2) \) - preinvex function on the co-ordinates, then the following inequality holds:
have preinvex function on the co-ordinates, then the following inequality holds:

\[
\begin{align*}
\frac{1}{\eta_1(b,a)\eta_2(d,c)} & \left| f \left( a + \frac{1}{2} \eta_1(b,a), c + \frac{1}{2} \eta_2(d,c) \right) 
\frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f \left( x, c + \frac{1}{2} \eta_2(d,c) \right) dx 
\frac{1}{\eta_2(d,c)} \int_c^{c+\eta_2(d,c)} f \left( a + \frac{1}{2} \eta_1(b,a), y \right) dy 
\frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y) dxdy \right| 
\leq \frac{1}{4(p+1)^{\frac{q}{p}}} \left( \frac{\partial^2 f}{\partial t_2 \partial t_1} (b,c) \right)^q \int_0^1 h_1(1-t_1) dt_1 \int_0^1 h_2(t_2) dt_2 
\end{align*}
\]

which completes the proof.

**Theorem 6.** Let \( f: \Delta \to \mathbb{R} \) be a partial differentiable on \( \Delta \) if \( \frac{\partial^2 f}{\partial t_2 \partial t_1} \geq 1 \), then a \((h_1, h_2)\)-preinvex function on the co-ordinates, then the following inequality holds:

\[
\begin{align*}
\frac{1}{\eta_1(b,a)\eta_2(d,c)} & \left| f \left( a + \frac{1}{2} \eta_1(b,a), c + \frac{1}{2} \eta_2(d,c) \right) 
\frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f \left( x, c + \frac{1}{2} \eta_2(d,c) \right) dx 
\frac{1}{\eta_2(d,c)} \int_c^{c+\eta_2(d,c)} f \left( a + \frac{1}{2} \eta_1(b,a), y \right) dy 
\frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y) dxdy \right| 
\leq \frac{1}{16} \left( \frac{1}{4} \right)^{\frac{q}{p}} \left[ \frac{\partial^2 f}{\partial t_2 \partial t_1} (a,c)^q + \frac{\partial^2 f}{\partial t_2 \partial t_1} (a,d)^q 
\frac{\partial^2 f}{\partial t_2 \partial t_1} (b,c)^q + \frac{\partial^2 f}{\partial t_2 \partial t_1} (b,d)^q \right] 
\times \int_0^1 t_2(h_2(1-t_2) + h_2(t_2)) dt_2 
\end{align*}
\]
Proof. By using the Lemma 2, well known power mean inequality for double integrals and the \((h_1, h_2)\) - preinvexity of \(f\) on the co-ordinates, the one has:

\[
\begin{align*}
\frac{1}{\eta_1(b,a)\eta_2(d,c)} & \int_a^c f \left( x, c + \frac{1}{2} \eta_2(d,c) \right) dx \\
& - \frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f \left( a + \frac{1}{2} \eta_1(b,a), c \right) dx \\
& - \frac{1}{\eta_2(d,c)} \int_c^{c+\eta_2(d,c)} f \left( a, c + \frac{1}{2} \eta_2(d,c) \right) dy \\
& + \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^c \int_c^{c+\eta_2(d,c)} f(x,y) \, dx \, dy \\
& \leq \left( \int_0^1 |p(t_1)q(t_2)| \, dt_1 \, dt_2 \right)^{1-\frac{1}{q}} \\
& \times \left( \int_0^1 |p(t_1)q(t_2)| \, \left| \frac{\partial^2 f}{\partial t_2 \partial t_1} (a + t_1 \eta_1(b,a), c + t_2 \eta_2(d,c)) \right|^q \, dt_1 \, dt_2 \right)^{\frac{1}{q}} \\
& \leq \left( \frac{1}{16} \right)^{1-\frac{1}{q}} \left\{ \left[ \frac{\partial^2 f}{\partial t_2 \partial t_1} (a,c) \right]^q + \left[ \frac{\partial^2 f}{\partial t_2 \partial t_1} (a,d) \right]^q \right. \\
& \quad + \left[ \frac{\partial^2 f}{\partial t_2 \partial t_1} (b,c) \right]^q + \left[ \frac{\partial^2 f}{\partial t_2 \partial t_1} (b,d) \right]^q \left\} \\
& \times \int_0^1 t_2 (h_2(1-t_2) + h_2(t_2)) \, dt_2 \\
& \quad \times \left( \int_0^1 t_1 (h_1(1-t_1) + h_1(t_1)) \, dt_1 \right)^{\frac{1}{q}}
\end{align*}
\]

which completes the proof.

REFERENCES


