A Study on Two Types of Multiple Integrals

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Abstract This paper takes the mathematical software Maple as the auxiliary tool to study two types of multiple integrals. We can obtain the closed forms of these two types of multiple integrals. On the other hand, we propose two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

Keywords Multiple Integrals, Closed Forms, Maple

1. Introduction

The multiple integral problem is closely related with probability theory and quantum field theory, and can refer to [1-2]. For this reason, the evaluation and numerical calculation of multiple integrals is important. In this study, we evaluate the following two types of n -tuple integrals

$$\int_{r_n}^{s_n} \cdots \int_{r_1}^{s_1} \exp\left(\sum_{k=1}^n a_k x_k\right) \cos\left(\sum_{k=1}^n b_k x_k + c\right) dx_1 \cdots dx_n \quad (1)$$

$$\int_{r_n}^{s_n} \cdots \int_{r_1}^{s_1} \exp\left(\sum_{k=1}^n a_k x_k\right) \sin\left(\sum_{k=1}^n b_k x_k + c\right) dx_1 \cdots dx_n \quad (2)$$

Where n is any positive integer, c, a_k, b_k, r_k, s_k are real numbers, $a_k^2 + b_k^2 \neq 0$ for $k = 1, \dots, n$. We can obtain the closed forms of these two types of multiple integrals; these are the major results of this study (i.e., Theorems 1 and 2). Moreover, we obtain some corollaries from these two theorems. For the study of related multiple integral problems can refer to [3-11]. In addition, we provide some multiple integrals to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these

solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce some notations and some properties used in this paper.

2.1. Notations

2.1.1. Let $z = a + ib \in C$, where $i = \sqrt{-1}$, a, b are real numbers, and C is the set of all complex numbers. We denote a the real part of z by $\text{Re}(z)$, and b the imaginary part of z by $\text{Im}(z)$.

2.1.2. $\prod_{k=1}^n p_k = p_1 p_2 \cdots p_n$, where $p_k \in C$ for all $k = 1, \dots, n$.

2.2. Euler's formula ([12])

$e^{ix} = \cos x + i \sin x$, where x is any real number.

2.3. Fundamental theorem of calculus ([13])

Let G be open in C , and let γ be a rectifiable path in G with initial and end points α and β respectively. If $f : G \rightarrow C$ is a continuous function with a primitive $F : G \rightarrow C$, then $\int_{\gamma} f = F(\beta) - F(\alpha)$.

(F is a primitive of f if $\frac{d}{dz} F(z) = f(z)$.)

Before deriving the major results in this study, we need a lemma.

2.4. Lemma A

Suppose a, b, r, s are real numbers, $a^2 + b^2 \neq 0$. Then the integral

$$\int_r^s \exp[(a+ib)x]dx = \frac{\exp[(a+ib)s] - \exp[(a+ib)r]}{a+ib} \quad (3)$$

2.4.1. Proof

Because $\frac{d}{dz} \left[\frac{1}{\lambda} \exp(\lambda z) \right] = \exp(\lambda z)$,

where $\lambda, z \in C$, $\lambda = a+ib \neq 0$. Let γ be a line in x -axis with initial and end points r and s respectively. Then we have

$$\begin{aligned} & \int_r^s \exp[(a+ib)x]dx \\ &= \int_{\gamma} \exp(\lambda z) dz \\ &= \frac{1}{\lambda} [\exp(\lambda s) - \exp(\lambda r)] \quad (\text{Using 2.3}) \\ &= \frac{\exp[(a+ib)s] - \exp[(a+ib)r]}{a+ib} \end{aligned}$$

The following is the first major result in this study, we determine the closed forms of the multiple integral (1).

2.5. Theorem 1

If n is any positive integer, and c, a_k, b_k, r_k, s_k are real numbers, $a_k^2 + b_k^2 \neq 0$ for $k = 1, \dots, n$. Then the n -tuple integral

$$\begin{aligned} & \int_{r_n}^{s_n} \cdots \int_{r_1}^{s_1} \exp\left(\sum_{k=1}^n a_k x_k\right) \cos\left(\sum_{k=1}^n b_k x_k + c\right) dx_1 \cdots dx_n \\ &= \operatorname{Re} \left[\exp(ic) \cdot \prod_{k=1}^n \frac{\exp[(a_k + ib_k)s_k] - \exp[(a_k + ib_k)r_k]}{a_k + ib_k} \right] \end{aligned} \quad (4)$$

2.5.1. Proof

$$\begin{aligned} & \int_{r_n}^{s_n} \cdots \int_{r_1}^{s_1} \exp\left(\sum_{k=1}^n a_k x_k\right) \cos\left(\sum_{k=1}^n b_k x_k + c\right) dx_1 \cdots dx_n \\ &= \int_{r_n}^{s_n} \cdots \int_{r_1}^{s_1} \operatorname{Re} \left[\exp\left(\sum_{k=1}^n a_k x_k\right) \exp i \left(\sum_{k=1}^n b_k x_k + c \right) \right] dx_1 \cdots dx_n \\ & \quad (\text{By Euler's formula}) \\ &= \operatorname{Re} \left[\exp(ic) \cdot \int_{r_n}^{s_n} \cdots \int_{r_1}^{s_1} \exp\left(\sum_{k=1}^n (a_k + ib_k)x_k\right) dx_1 \cdots dx_n \right] \\ &= \operatorname{Re} \left[\exp(ic) \cdot \prod_{k=1}^n \int_{r_k}^{s_k} \exp[(a_k + ib_k)x_k] dx_k \right] \\ &= \operatorname{Re} \left[\exp(ic) \cdot \prod_{k=1}^n \frac{\exp[(a_k + ib_k)s_k] - \exp[(a_k + ib_k)r_k]}{a_k + ib_k} \right] \\ & \quad (\text{By Lemma A}) \end{aligned}$$

By Theorem 1, we immediately have the following result.

2.6. Corollary 1

Suppose n is any positive integer, c, a_k, b_k are real numbers, $a_k < 0$ for $k = 1, \dots, n$. Then the n -tuple improper integral

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \exp\left(\sum_{k=1}^n a_k x_k\right) \cos\left(\sum_{k=1}^n b_k x_k + c\right) dx_1 \cdots dx_n \\ &= (-1)^n \cdot \operatorname{Re} \left[\frac{\exp(ic)}{\prod_{k=1}^n (a_k + ib_k)} \right] \end{aligned} \quad (5)$$

In Corollary 1, taking $x_k = \ln y_k$, where $y_k \geq 1$ for $k = 1, \dots, n$. Then we obtain the following result.

2.7. Corollary 2

If the assumptions are the same as Corollary 1, then the n -tuple improper integral

$$\begin{aligned} & \int_1^\infty \cdots \int_1^\infty \prod_{k=1}^n y_k^{a_k-1} \cdot \cos\left(\sum_{k=1}^n b_k \ln y_k + c\right) dy_1 \cdots dy_n \\ &= (-1)^n \cdot \operatorname{Re} \left[\frac{\exp(ic)}{\prod_{k=1}^n (a_k + ib_k)} \right] \end{aligned} \quad (6)$$

The following is the second major result in this paper, we obtain the closed forms of the multiple integral (2).

2.8. Theorem 2

If the assumptions are the same as Theorem 1. Then the n -tuple integral

$$\begin{aligned} & \int_{r_n}^{s_n} \cdots \int_{r_1}^{s_1} \exp\left(\sum_{k=1}^n a_k x_k\right) \sin\left(\sum_{k=1}^n b_k x_k + c\right) dx_1 \cdots dx_n \\ &= \operatorname{Im} \left[\exp(ic) \cdot \prod_{k=1}^n \frac{\exp[(a_k + ib_k)s_k] - \exp[(a_k + ib_k)r_k]}{a_k + ib_k} \right] \end{aligned} \quad (7)$$

2.8.1. Proof

$$\begin{aligned} & \int_{r_n}^{s_n} \cdots \int_{r_1}^{s_1} \exp\left(\sum_{k=1}^n a_k x_k\right) \sin\left(\sum_{k=1}^n b_k x_k + c\right) dx_1 \cdots dx_n \\ &= \int_{r_n}^{s_n} \cdots \int_{r_1}^{s_1} \operatorname{Im} \left[\exp\left(\sum_{k=1}^n a_k x_k\right) \exp i \left(\sum_{k=1}^n b_k x_k + c \right) \right] dx_1 \cdots dx_n \\ & \quad (\text{By Euler's formula}) \end{aligned}$$

$$\begin{aligned}
&= \text{Im} \left[\exp(ic) \cdot \prod_{k=1}^n \int_{r_k}^{s_k} \exp[(a_k + ib_k)x_k] dx_k \right] \\
&= \text{Im} \left[\exp(ic) \cdot \prod_{k=1}^n \frac{\exp[(a_k + ib_k)s_k] - \exp[(a_k + ib_k)r_k]}{a_k + ib_k} \right] \\
&\quad \text{(By Lemma A)}
\end{aligned}$$

Using Theorem 2, we have the following result.

2.9. Corollary 3

If the assumptions are the same as Corollary 1, then the n -tuple improper integral

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \exp\left(\sum_{k=1}^n a_k x_k\right) \sin\left(\sum_{k=1}^n b_k x_k + c\right) dx_1 \cdots dx_n \\
&= (-1)^n \cdot \text{Im} \left[\frac{\exp(ic)}{\prod_{k=1}^n (a_k + ib_k)} \right] \quad (8)
\end{aligned}$$

In Corollary 3, let $x_k = \ln y_k$, where $y_k \geq 1$ for $k = 1, \dots, n$. Then the following result holds.

2.10. Corollary 4

If the assumptions are the same as Corollary 1, then the n -tuple improper integral

$$\begin{aligned}
&\int_1^\infty \cdots \int_1^\infty \prod_{k=1}^n y_k^{a_k-1} \cdot \sin\left(\sum_{k=1}^n b_k \ln y_k + c\right) dy_1 \cdots dy_n \\
&= (-1)^n \cdot \text{Im} \left[\frac{\exp(ic)}{\prod_{k=1}^n (a_k + ib_k)} \right] \quad (9)
\end{aligned}$$

3. Examples

In the following, for the two types of multiple integrals in this study, we provide two examples and use Theorems 1, 2 and Corollaries 1, 2, 3, 4 to determine the closed forms of these multiple integrals. On the other hand, we employ Maple to calculate the approximations of these multiple integrals and their solutions for verifying our answers.

3.1. Example 1 By Theorem 1, we obtain the double integral

$$\int_{\pi/2}^{2\pi} \int_{\pi/3}^{\pi} \exp(2x_1 - 3x_2) \cos\left(5x_1 + 4x_2 + \frac{\pi}{3}\right) dx_1 dx_2$$

$$\begin{aligned}
&= \text{Re} \left[\frac{\exp(i\frac{\pi}{3}) \cdot [\exp \pi(2 + 5i) - \exp \frac{\pi}{3}(2 + 5i)]}{(2 + 5i)(-3 + 4i)} \times \right. \\
&\quad \left. [\exp 2\pi(-3 + 4i) - \exp \frac{\pi}{2}(-3 + 4i)] \right] \\
&= \frac{(26 + 7\sqrt{3})[\exp(-4\pi) - \exp \frac{\pi}{2}] + 52[\exp(\frac{-16\pi}{3}) - \exp(\frac{-5\pi}{6})]}{1450} \quad (10)
\end{aligned}$$

We use Maple to verify the correctness of (10) as follows:

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>evalf(Doubleint(exp(2*x1-3*x2)*cos(5*x1+4*x2+Pi/3),
x1=Pi/3..Pi,x2=Pi/2..2*Pi),14);
-0.129096252409693097
>evalf(((26+7*sqrt(3))*(exp(-4*Pi)-exp(Pi/2))+52*(exp(-16
*Pi/3)-exp(-5*Pi/6)))/1450,18);
-0.129096252409693097

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Also, by Corollary 1, we obtain the triple improper integral

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \int_0^\infty \exp(-6x_1 - x_2 - 2x_3) \cos\left(x_1 + x_2 + 4x_3 - \frac{\pi}{4}\right) dx_1 dx_2 dx_3 \\
&= -\text{Re} \left[\frac{\exp(-i\frac{\pi}{4})}{(-6 + i)(-1 + i)(-2 + 4i)} \right] \\
&= \frac{\sqrt{2}}{185} \approx 0.0076443976344491 \quad (11)
\end{aligned}$$

On the other hand, using Corollary 2, the following double improper integral

$$\begin{aligned}
&\int_1^\infty \int_1^\infty y_1^{-2} y_2^{-4} \cdot \cos\left(5 \ln y_1 - 4 \ln y_2 + \frac{\pi}{3}\right) dy_1 dy_2 \\
&= \text{Re} \left[\frac{\exp(i\frac{\pi}{3})}{(-1 + 5i)(-3 - 4i)} \right] \\
&= \frac{23 - 11\sqrt{3}}{1300} \approx 0.003036491623 \quad (12)
\end{aligned}$$

3.2. Example 2

By Theorem 2, the double integral

$$\begin{aligned}
&\int_{\pi/2}^{\pi} \int_{\pi/4}^{2\pi} \exp(4x_1 + 2x_2) \sin\left(8x_1 - 4x_2 + \frac{3\pi}{4}\right) dx_1 dx_2 \\
&= \text{Im} \left[\frac{\exp(i\frac{3\pi}{4}) \cdot [\exp 2\pi(4 + 8i) - \exp \frac{\pi}{4}(4 + 8i)]}{(4 + 8i)(2 - 4i)} \times \right.
\end{aligned}$$

$$\begin{aligned}
& \left[\exp \pi(2-4i) - \exp \frac{\pi}{2}(2-4i) \right] \\
&= \frac{\sqrt{2}(\exp 8\pi - \exp \pi)(\exp 2\pi - \exp \pi)}{80} \\
&\approx 7.44737801177467992 \times 10^{11} \quad (13)
\end{aligned}$$

Also, by Corollary 3, the triple improper integral

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \int_0^\infty \exp(-x_1 - 3x_2 - x_3) \sin\left(2x_1 - x_2 + 3x_3 + \frac{\pi}{6}\right) dx_1 dx_2 dx_3 \\
&= -\operatorname{Im} \left[\frac{\exp(i\frac{\pi}{6})}{(-1+2i)(-3-i)(-1+3i)} \right] \\
&= \frac{2\sqrt{3}-1}{100} \approx 0.024641016151378 \quad (14)
\end{aligned}$$

By Corollary 4, we get the double improper integral

$$\begin{aligned}
& \int_1^\infty \int_1^\infty y_1^{-4} y_2^{-3} \cdot \sin\left(7\ln y_1 + 2\ln y_2 - \frac{3\pi}{4}\right) dy_1 dy_2 \\
&= \operatorname{Im} \left[\frac{\exp(-i\frac{3\pi}{4})}{(-3+7i)(-2+2i)} \right] \\
&= \frac{-3\sqrt{2}}{232} \approx -0.01828724439151 \quad (15)
\end{aligned}$$

4. Conclusion

As mentioned, the multiple integral problem is important in probability theory and quantum field theory. In this study, we propose a new technique to solve some types of multiple integrals, and we hope this method can be applied in mathematical statistics or quantum physics. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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