Continuous Generalized Hankel-Type Wavelet Transformation

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Abstract In this paper, the generalized Hankel-type wavelet transformation is developed. Using the developed theory of generalized Hankel-type convolution, the generalized Hankel-type translation is introduced. Properties of the kernel $D_{\mu,\nu}(x,y,z)$ are developed in the study. Using the properties of kernel the generalized Hankel-type wavelet transformation is defined. The existence of the generalized Hankel-type wavelet transformation is given by a theorem. The boundedness and inversion formula for the generalized Hankel-type wavelet transformation is obtained. A basic wavelet which defines continuous generalized Hankel-type wavelet transformation, its admissibility conditions and the wavelet to the function is proved. Examples have been shown to explain the studied continuous generalized Hankel-type wavelet transformation.
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1. Introduction

Malgonde [1] investigated the following generalized Hankel-type transformation

$$F_1(t) = (F_{1,\mu,\nu}f)(t) = \nu t^{-1} \int_0^\infty (xt)^\nu J_\mu[(xt)^\nu] f(x) dx, \quad (1)$$

$$\lim_{R \rightarrow \infty} \nu y_0^{-1} \int_0^R (x_0 y)^\nu J_\mu[(x_0 y)^\nu] F_1(y) dy = \frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)]. \quad (5)$$

If $f(x)x^{-\nu-\mu}$ and $F_2(y)y^{\mu+\nu-1}$ are in $L_1(0, \infty)$, for

$$F_1(t) = (F_{1,\mu,\nu}f)(t) = \nu t^{-1} \int_0^\infty (xt)^\nu J_\mu[(xt)^\nu] f(x) dx, \quad (6)$$

and

$J_\mu(x)$ being the Bessel function of the first kind of order $\mu \geq -1/2$.

Let $L_p(0, \infty)$, $1 \leq p \leq \infty$, be defined as the space of real measurable function ϕ on $(0, \infty)$ for which

$$\|\phi\|_{\mu,\nu,p} = \left(\int_0^\infty |x^{\mu\nu-\nu} \phi(x)|^p \frac{dx}{x} \right)^{1/p}, \quad 1 \leq p < \infty \quad (2)$$

$$\|\phi\|_\infty = \text{ess sup}_{0 < x < \infty} |x^{\mu\nu-\nu} \phi(x)| < \infty. \quad (3)$$

For each $\phi \in L_1(0, \infty)$, generalized Hankel-type transformation of ϕ is defined by

$$\hat{\phi}(x) = \nu t^{-1} \int_0^\infty (xt)^\nu J_\mu[(xt)^\nu] \phi(t) dt, \quad 0 < t < \infty. \quad (4)$$

From [1], it is known that $\hat{\phi}(x)$ is bounded and continuous on $(0, \infty)$ and $\|\hat{\phi}(x)\|_\infty \leq \|\phi\|_1$.

If $f(x)$ is of bounded variation into a neighborhood of the point $x = x_0 > 0$, $\mu \geq -1/2$ and the integral $\int_0^\infty |f(x)|x^{\nu/2}$ exists, then the inversion formula in [2] is given by

$$F_2(t) = (F_{2,\mu,\nu} g(x))(t) = \nu t^{-1} \int_0^\infty (xt)^\nu J_\mu \left[(xt)^\nu \right] g(x) dx ,$$

for $\mu \geq -1/2$,

the following mixed Parseval formula holds for F_1 -transformation by [2];

$$\int_0^\infty f(x) g(x) dx = \int_0^\infty F_1(y) F_2(y) dy.$$

If $\phi(x)$ and $\psi(x)$ are in L_1 , then the following Parseval's formula holds

$$\int_0^\infty \hat{\phi}(t) \hat{\psi}(t) dt = \int_0^\infty \phi(x) \psi(x) dy. \quad (7)$$

To define the generalized Hankel-type Convolution, there is a need to introduce generalized Hankel-type translation.

$$D_{\mu,\nu}(x, y, z) = \int_0^\infty t^{-\mu\nu-\nu} \nu t^{-1} (xt)^\nu J_\mu \left[(xt)^\nu \right] \nu t^{-1} (yt)^\nu J_\mu \left[(yt)^\nu \right] \nu t^{-1} (zt)^\nu J_\mu \left[(zt)^\nu \right] dt \quad (8)$$

is defined.

Properties of the kernel $D_{\mu,\nu}(x, y, z)$:

Following [3] properties are established:

i) For $0 < x, y < \infty$ and $0 \leq t < \infty$, we have

$$\int_0^\infty \nu t^{\mu\nu+\nu} z^{-1} (zt)^\nu J_\mu \left[(zt)^\nu \right] D_{\mu,\nu}(x, y, z) dz = \nu^2 (xy)^{-1} (xt)^\nu J_\mu \left[(xt)^\nu \right] (yt)^\nu J_\mu \left[(yt)^\nu \right] \quad (9)$$

Proof: $D_{\mu,\nu}(x, y, z) = (\nu)^3 \int_0^\infty t^{-\mu\nu-\nu} z^{-1} (zt)^\nu J_\mu \left[(zt)^\nu \right] \left[x^{-1} (xt)^\nu J_\mu \left[(xt)^\nu \right] y^{-1} (yt)^\nu J_\mu \left[(yt)^\nu \right] \right] dz$

$$= \int_0^\infty \nu t^{-\mu\nu-\nu} z^{-1} \left[\nu^2 (xy)^{-1} (xt)^\nu J_\mu \left[(xt)^\nu \right] (yt)^\nu J_\mu \left[(yt)^\nu \right] \right] (zt)^\nu J_\mu \left[(zt)^\nu \right] dz$$

$$= t^{-\mu\nu-\nu} F_{1,\mu,\nu} \left\{ \nu^2 (xy)^{-1} (xt)^\nu J_\mu \left[(xt)^\nu \right] (yt)^\nu J_\mu \left[(yt)^\nu \right] \right\}$$

$$F_{1,\mu,\nu}^{-1} \left\{ t^{\mu\nu+\nu} D_{\mu,\nu}(x, y, z) \right\} = \nu^2 (xy)^{-1} (xt)^\nu J_\mu \left[(xt)^\nu \right] (yt)^\nu J_\mu \left[(yt)^\nu \right].$$

Applying the inversion formula of generalized Hankel-type transformation to (8)

$$\int_0^\infty \nu t^{\mu\nu+\nu} z^{-1} (zt)^\nu J_\mu \left[(zt)^\nu \right] D_{\mu,\nu}(x, y, z) dz = \nu^2 (xy)^{-1} (xt)^\nu J_\mu \left[(xt)^\nu \right] (yt)^\nu J_\mu \left[(yt)^\nu \right] \quad (10)$$

and hence the result. In particular, taking $t = 0$, gives

$$\int_0^\infty \nu t^{\mu\nu+\nu} z^{-1} (zt)^\nu J_\mu \left[(zt)^\nu \right] D_{\mu,\nu}(x, y, z) dz = 1 \quad (11)$$

i.e. for which $x, y > 0$, $D_{\mu,\nu}(x, y, z)$ belongs to $L^1_{0,\nu,\mu}(0, \infty)$.

$$\text{iii) } 0 < x, y, z < \infty, D_{\mu,\nu}(x, y, z) \geq 0. \quad (12)$$

$$\text{iv) } D_{\mu,\nu}(x, y, z) = D_{\mu,\nu}(y, x, z) = D_{\mu,\nu}(z, x, y) = \dots \quad (13)$$

The generalized Hankel-type translation T_y of $\phi \in L_p(0, \infty)$, $1 \leq p \leq \infty$, is defined by

$$T_y \phi(x) = \phi(x, y) = \int_0^\infty \phi(z) D_{\mu, \nu}(x, y, z) dz, 0 < x, y < \infty.$$

The map $y \rightarrow T_y \phi$ is continuous from $(0, \infty)$ into $(0, \infty)$. Let $p, q, r \in [1, \infty)$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. The generalized Hankel-type convolution of $\phi \in L_p(0, \infty)$ and $\psi \in L_p(0, \infty)$ is defined by

$$(\phi \# \psi)(x) = \int_0^\infty \phi(x, y) \psi(y) dy. \quad (14)$$

In [7] the integral is convergent for almost all $x, 0 < x < \infty$ and

$$\|\phi \# \psi\|_r \leq \|\phi\|_p \|\psi\|_q. \quad (15)$$

Moreover, $p = \infty$, then $(\phi \# \psi)(x)$ is defined for all $x, 0 < x < \infty$ and is continuous.

If $\phi, \psi \in L_1(0, \infty)$, then

$$(\phi \# \psi) \wedge(t) = \hat{\phi}(t) \hat{\psi}(t), 0 \leq t < \infty. \quad (16)$$

In this paper, in terms of the aforesaid generalized Hankel-type translation T_y and dilation D_a defined by

$$D_{\mu, \nu, a} \phi(x, y) = a^{-2\mu\nu-\nu} \phi(x/a, y/a), \quad (17)$$

is a continuous generalized Hankel-type wavelet transformation. Its continuity and boundedness properties are established. An inversion formula is obtained.

2. Continuous Generalized Hankel-Type Wavelet Transformation

Let $\psi \in L_p(0, \infty), 1 \leq p < \infty$ be given. For $b \geq 0$ and $a > 0$ define the generalized Hankel-type wavelet transformation

$$\begin{aligned} \psi_{b,a}(x) &= D_{\mu, \nu, a} T_y \psi(x) = D_{\mu, \nu, a} \psi(b, x) = a^{-2\mu\nu-\nu} \psi(b/a, x/a) \\ &= a^{-2\mu\nu-\nu} \int_0^\infty D_{\mu, \nu}(b/a, x/a, z) \psi(z) dz, \end{aligned} \quad (18)$$

the integral being convergent by virtue of (15).

Using the wavelet $\psi_{b,a}$, the generalized Hankel-type wavelet transformation is defined.

$$\begin{aligned} H_{1, \nu, \mu}(b, a) &= (H_{1, \nu, \mu, \psi} f)(b, a) \\ &= \langle f(t), \psi_{b,a}(t) \rangle \\ &= \int_0^\infty f(t) \overline{\psi_{b,a}(t)} dt \end{aligned} \quad (19)$$

$$= a^{-2\mu\nu-\nu} \int_0^\infty \int_0^\infty f(t) \overline{\psi(z)} D_{\mu, \nu}(b/a, t/a, z) dz dt \quad (20)$$

provided the integral is convergent. The continuity of the generalized Hankel-type wavelet follows from the boundedness property of the generalized Hankel-type translation [5].

Lemma 1. Let $\psi \in L_p(0, \infty), 1 \leq p < \infty$. Then for $y \geq 0$, the map $y \rightarrow T_y f$ is continuous from $L_p(0, \infty)$ into

$L_p(0, \infty)$. The function $\psi_{b,a}$ is defined almost everywhere on $[0, \infty)$, and

$$\|\psi_{b,a}(x)\|_p \leq a^{(2\mu\nu+\nu)(1/p-1)} \|\psi\|_p. \quad (21)$$

The existence of the generalized Hankel-type wavelet transformation is given by the following theorem.

Theorem 2. Let $f \in L_p(0, \infty)$ and $\psi \in L_q(0, \infty)$ with $1 \leq p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$;

$H_{1,\nu,\mu}(b, a) = (H_{1,\nu,\mu,\psi} f)(b, a)$ be the continuous wavelet transform (19). Then

$(H_{1,\nu,\mu} f)(b, a)$ is continuous on $(0, \infty) \times (0, \infty)$,

$$\left\| \left((H_{1,\nu,\mu,\psi} f)(b, a) f \right)(b, a) \right\|_r \leq a^{2\mu\nu+\nu} \|f\|_p \|\psi\|_q, \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1, 1 \leq p, q, r < \infty,$$

$$\left\| \left((H_{1,\nu,\mu,\psi} f)(b, a) f \right)(b, a) \right\|_\infty \leq a^{(2\mu\nu+\nu)\left(\frac{1}{q}-1\right)} \|f\|_p \|\psi\|_q, \frac{1}{p} + \frac{1}{q} = 1.$$

Proof.

Let (b_0, a_0) be an arbitrary but fixed point in $(0, \infty) \times (0, \infty)$. Then by Hölder's inequality,

$$\begin{aligned} & \left| (H_{1,\nu,\mu} f)(b, a) - (H_{1,\nu,\mu} f)(b_0, a_0) \right| \\ & \leq a^{-2\mu\nu-\nu} \int_0^\infty \int_0^\infty |f(t)\psi(z)| \left[D_{\mu,\nu}(b/a, t/a, z) - D_{\mu,\nu}(b_0/a_0, t/a_0, z) \right] dt dz \\ & \leq a^{-2\mu\nu-\nu} \left[\int_0^\infty \int_0^\infty |f(t)|^p \left| D_{\mu,\nu}(b/a, t/a, z) - D_{\mu,\nu}(b_0/a_0, t/a_0, z) \right| dt dz \right]^{1/p} \\ & \quad \times \left[\int_0^\infty \int_0^\infty |\psi(z)|^q \left| D_{\mu,\nu}(b/a, t/a, z) - D_{\mu,\nu}(b_0/a_0, t/a_0, z) \right| dt dz \right]^{1/q} \end{aligned}$$

Since by (9) $\int_0^\infty \left| D_{\mu,\nu}(b/a, t/a, z) - D_{\mu,\nu}(b_0/a_0, t/a_0, z) \right| dt \leq 2$, by dominated convergence theorem and

continuity of $D_{\mu,\nu}(b/a, t/a, z)$ in the variables b and a , we have $\lim_{\substack{b \rightarrow b_0 \\ a \rightarrow a_0}} \left| (H_{1,\nu,\mu} f)(b, a) - (H_{1,\nu,\mu} f)(b_0, a_0) \right| = 0$.

This proves that $H(b, a)$ is continuous on $(0, \infty) \times (0, \infty)$.

Inequality (15) proves

$$\left\| \left((H_{1,\nu,\mu,\psi} f)(b, a) f \right)(b, a) \right\|_r \leq a^{2\mu\nu+\nu} \|f\|_p \|\psi\|_q, \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1, 1 \leq p, q, r < \infty.$$

It can be proved using Hölder's inequality.

3. An inversion Formula

In this section, it is shown that the function f can be recovered from its wavelet transform when the wavelet ψ satisfies admissibility condition.

Theorem 3. Let $\psi \in L^2(\mathbb{R}_+)$ be a basic wavelet which defines generalized Hankel-type wavelet transformation (18). Then, for

$$A_\psi = \int_0^\infty w^{-2\mu\nu-\nu} |\hat{\psi}(w)|^2 dw > 0, \quad (22)$$

we have

$$\int_0^\infty \int_0^\infty \left((H_{1,\nu,\mu,\psi} f)(b,a) f)(b,a) \overline{\left((H_{1,\nu,\mu,\psi} f)(b,a) g)(b,a) \right)} a^{-2\mu\nu-\nu} da db = A_\psi \langle f, g \rangle \quad (23)$$

for all $f, g \in L^2(\mathbb{R}_+)$.

Proof. In view of representation (8) we have

$$\begin{aligned} & (H_{1,\nu,\mu,\psi} f)(b,a) \\ &= \int_0^\infty \int_0^\infty f(t) \overline{\psi(z)} D_{\mu,\nu}(b/a, t/a, z) dz dt \\ &= a^{-2\mu\nu-\nu} \int_0^\infty \int_0^\infty \hat{f}(x/a) \overline{\psi(z)} \left[\left\{ \nu \left(\frac{b}{a} \right)^{-1} \left(\frac{xb}{a} \right)^\nu J_\mu \left[\left(\frac{xb}{a} \right)^\nu \right] \right\} \right] dx dz \\ & \quad \left[\left\{ \nu z^{-1} (zax)^\nu J_\mu \left[(zax)^\nu \right] \right\} \right] \\ &= a^{-2\mu\nu-\nu} \int_0^\infty \hat{f}(x/a) \overline{\hat{\psi}(x)} \left\{ \nu \left(\frac{b}{a} \right)^{-1} \left(\frac{xb}{a} \right)^\nu J_\mu \left[\left(\frac{xb}{a} \right)^\nu \right] \right\} dx \\ &= \int_0^\infty \hat{f}(u) \overline{\hat{\psi}(au)} \left\{ \nu b^{-1} (bu)^\nu J_\mu \left[(bu)^\nu \right] \right\} du \\ &= \left(\hat{f}(u) \overline{\hat{\psi}(au)} \right) \wedge (b). \end{aligned} \quad (24)$$

Parseval identity (7) yields

$$\begin{aligned} & \int_0^\infty (H_{1,\nu,\mu,\psi} f)(b,a) \overline{(H_{1,\nu,\mu,\psi} f)(b,a)} db \\ &= \int_0^\infty \left(\hat{f}(u) \overline{\hat{\psi}(au)} \right) \wedge (b) \overline{\left(\hat{g}(u) \overline{\hat{\psi}(au)} \right) \wedge (b)} db \\ &= \int_0^\infty \left(\hat{f}(u) \overline{\hat{\psi}(au)} \right) \overline{\left(\hat{g}(u) \overline{\hat{\psi}(au)} \right)} du. \end{aligned}$$

Now multiplying by $a^{-2\mu\nu-\nu} da$ and integrating, we get

$$\begin{aligned}
 & \int_0^\infty \int_0^\infty (H_{1,\nu,\mu,\psi} f)(b, a) \overline{(H_{1,\nu,\mu,\psi} f)(b, a)} db a^{-2\mu\nu-\nu} da \\
 &= \int_0^\infty \int_0^\infty (\hat{f}(u) \overline{\hat{\psi}(au)}) \overline{(\hat{g}(u) \overline{\hat{\psi}(au)})} du a^{-2\mu\nu-\nu} da \\
 &= \int_0^\infty \hat{f}(u) \overline{\hat{g}(u)} du \int_0^\infty \hat{\psi}(au) \overline{\hat{\psi}(au)} a^{-2\mu\nu-\nu} da \\
 &= \int_0^\infty \hat{f}(u) \overline{\hat{g}(u)} du \int_0^\infty |\hat{\psi}(au)|^2 a^{-2\mu\nu-\nu} da \\
 &= \int_0^\infty \hat{f}(u) \overline{\hat{g}(u)} du \int_0^\infty |\hat{\psi}(w)|^2 w^{-2\mu\nu-\nu} dw \\
 &= A_\nu \langle f, g \rangle.
 \end{aligned}$$

Notice that admissibility condition (22) requires that $\hat{\psi}(0) = 0$. If $\hat{\psi}$ is continuous then from (3) it follows that

$$\int_0^\infty \psi(x) dx = 0.$$

This justifies the wavelet to the function.

Now considering

$$\psi\left(\frac{b}{a}, \frac{t}{a}\right) = \psi(x, y) \text{ by putting } \frac{b}{a} = x \text{ and } \frac{t}{a} = y. \text{ Then } \psi(x, y) = \int_0^\infty \psi(z) D_{\mu,\nu}(b/a, t/a, z) dz.$$

Since

$$D_{\mu,\nu}(x, y, z) = \int_0^\infty \xi^{-\mu\nu-\nu} \left[\nu x^{-1} (x\xi)^\nu J_\mu[(x\xi)^\nu] \nu y^{-1} (y\xi)^\nu J_\mu[(y\xi)^\nu] \nu z^{-1} (z\xi)^\nu J_\mu[(z\xi)^\nu] d\xi \right].$$

Substituting,

$$\begin{aligned}
 \psi(x, y) &= \int_0^\infty \psi(z) \left[\int_0^\infty \xi^{-\mu\nu-\nu} \left[\nu x^{-1} (x\xi)^\nu J_\mu[(x\xi)^\nu] \nu y^{-1} (y\xi)^\nu J_\mu[(y\xi)^\nu] \nu z^{-1} (z\xi)^\nu J_\mu[(z\xi)^\nu] d\xi \right] dz \right] \\
 &= \int_0^\infty \left(\int_0^\infty \psi(z) \nu z^{-1} (z\xi)^\nu J_\mu[(z\xi)^\nu] dz \right) \xi^{-\mu\nu-\nu} \left\{ \nu x^{-1} (x\xi)^\nu J_\mu[(x\xi)^\nu] \nu y^{-1} (y\xi)^\nu J_\mu[(y\xi)^\nu] \right\} d\xi \\
 &= \int_0^\infty \xi^{-\mu\nu-\nu} \left\{ \nu x^{-1} (x\xi)^\nu J_\mu[(x\xi)^\nu] \nu y^{-1} (y\xi)^\nu J_\mu[(y\xi)^\nu] \right\} (F_{1,\mu,\nu}\psi)(\xi) d\xi.
 \end{aligned}$$

Substituting $\frac{b}{a} = x$ and $\frac{t}{a} = y$.

$$\psi\left(\frac{b}{a}, \frac{t}{a}\right) = \int_0^\infty \xi^{-\mu\nu-\nu} \left\{ \nu \left(\frac{b}{a}\right)^{-1} \left(\frac{b\xi}{a}\right)^\nu J_\mu\left[\left(\frac{b\xi}{a}\right)^\nu\right] \nu \left(\frac{t}{a}\right)^{-1} \left(\frac{t\xi}{a}\right)^\nu J_\mu\left[\left(\frac{t\xi}{a}\right)^\nu\right] \right\} (F_{1,\mu,\nu}\psi)(\xi) d\xi.$$

$$\begin{aligned}
& (H_{1,\nu,\mu,\psi}f)(b,a) \\
&= a^{-2\mu\nu-\nu} \int_0^\infty \psi\left(\frac{b}{a}, \frac{t}{a}\right) f(t) dt \\
&= a^{-2\mu\nu-\nu} \int_0^\infty \int_0^\infty \xi^{-\mu\nu-\nu} \left\{ \begin{array}{l} \nu \left(\frac{b}{a}\right)^{-1} \left(\frac{b\xi}{a}\right)^\nu J_\mu \left[\left(\frac{b\xi}{a}\right)^\nu\right] \\ \nu \left(\frac{t}{a}\right)^{-1} \left(\frac{t\xi}{a}\right)^\nu J_\mu \left[\left(\frac{t\xi}{a}\right)^\nu\right] \end{array} \right\} (F_{1,\mu,\nu}\psi)(\xi) d\xi \times f(t) dt. \\
&= a^{-2\mu\nu-\nu} \int_0^\infty \nu \beta \left(\frac{b}{a}\right)^{-1} \left(\frac{b\xi}{a}\right)^\nu J_\mu \left[\left(\frac{b\xi}{a}\right)^\nu\right] \left(\int_0^\infty \nu \left(\frac{t}{a}\right)^{-1} \left(\frac{t\xi}{a}\right)^\nu J_\mu \left[\left(\frac{t\xi}{a}\right)^\nu\right] f(t) dt \right) \\
&\quad \times \xi^{-\mu\nu-\nu} (F_{1,\mu,\nu}\psi)(\xi) d\xi \\
&= a^{-2\mu\nu-\nu} \int_0^\infty \nu \beta \left(\frac{b}{a}\right)^{-1} \left(\frac{b\xi}{a}\right)^\nu J_\mu \left[\left(\frac{b\xi}{a}\right)^\nu\right] (F_{1,\mu,\nu}f)\left(\frac{\xi}{a}\right) \times \xi^{-\mu\nu-\nu} (F_{1,\mu,\nu}\psi)(\xi) d\xi.
\end{aligned}$$

By substituting $\frac{\xi}{a} = x; d\xi = adx$, the continuous generalized Hankel-type wavelet transform can be written as

$$\begin{aligned}
(H_{1,\nu,\mu,\psi}f)(b,a) &= a^{-2\mu\nu-\nu} \int_0^\infty \nu \left(\frac{b}{a}\right)^{-1} \left(\frac{b\xi}{a}\right)^\nu J_\mu \left[\left(\frac{b\xi}{a}\right)^\nu\right] (F_{1,\mu,\nu}f)\left(\frac{\xi}{a}\right) \times \xi^{-\mu\nu-\nu} (F_{1,\mu,\nu}\psi)(\xi) d\xi \\
&= a^{-2\mu\nu-\nu} \int_0^\infty \nu \left(\frac{b}{a}\right)^{-1} (bx)^\nu J_\mu \left[(bx)^\nu\right] (F_{1,\mu,\nu}f)(x) \times (ax)^{-\mu\nu-\nu} (F_{1,\mu,\nu}\psi)(ax) adx \\
&= a^{-2\mu\nu-\nu} a^{-\mu\nu-\nu+3/2} \int_0^\infty \nu b^{-1} (bx)^\nu J_\mu \left[(bx)^\nu\right] (F_{1,\mu,\nu}f)(x) x^{-\mu\nu-\nu} (F_{1,\mu,\nu}\psi)(ax) dx \\
&= a^{-2\mu\nu-\nu} a^{-\mu\nu-\nu+\frac{3}{2}} \int_0^\infty \nu b^{-1} (bx)^\nu J_\mu \left[(bx)^\nu\right] (F_{1,\mu,\nu}f)(x) x^{-\mu\nu-\nu} (F_{1,\mu,\nu}\psi)(ax) dx \\
&= a^{-3\mu\nu-2\nu+\frac{3}{2}} \int_0^\infty \nu b^{-1} (bx)^\nu J_\mu \left[(bx)^\nu\right] (F_{1,\mu,\nu}f)(x) x^{-\mu\nu-\nu} (F_{1,\mu,\nu}\psi)(ax) dx.
\end{aligned}$$

Example 1. Assume that $f(t) = t^{-\tau}; 0 < \tau < (2\mu+1)\nu$.

Then the generalized Hankel-type wavelet transforms of $t^{-\tau}$ is given by

$$\begin{aligned}
& (H_{1,\nu,\mu,\psi}(t^{-\tau}))(b,a) \\
&= a^{-2\mu\nu-\nu} \int_0^\infty t^{-\tau} \psi\left(\frac{b}{a}, \frac{t}{a}\right) dt \\
&= a^{-2\mu\nu-\nu} \int_0^\infty \psi(z) \left(\int_0^\infty t^{-\tau} D_{\mu,\nu}(b/a, t/a, z) dt \right) dz
\end{aligned} \tag{25}$$

Since

$$D_{\mu,\nu}(x,y,z) = \int_0^\infty \xi^{-\mu\nu-\nu} \left[\nu x^{-1} (x\xi)^\nu J_\mu \left[(x\xi)^\nu\right] \nu y^{-1} (y\xi)^\nu J_\mu \left[(y\xi)^\nu\right] \nu z^{-1} (z\xi)^\nu J_\mu \left[(z\xi)^\nu\right] d\xi \right]$$

$$x = \frac{b}{a}; y = \frac{t}{a};$$

$$D_{\mu,\nu}(b/a, t/a, z) = \int_0^\infty \xi^{-\mu\nu-\nu} \left[\begin{array}{l} \nu \left(\frac{b}{a}\right)^{-1} \left(\frac{b}{a}\xi\right)^\nu J_\mu \left[\left(\frac{b}{a}\xi\right)^\nu\right] \nu \left(\frac{t}{a}\right)^{-1} \left(\frac{t}{a}\xi\right)^\nu J_\mu \left[\left(\frac{t}{a}\xi\right)^\nu\right] \\ \nu z^{-1} (z\xi)^\nu J_\mu [(z\xi)^\nu] d\xi \end{array} \right]$$

$$\xi = ax; d\xi = adx.$$

$$D_{\mu,\nu}(b/a, t/a, z) = \int_0^\infty (ax)^{-\mu\nu-\nu} \left[\begin{array}{l} \nu \left(\frac{b}{a}\right)^{-1} \left(\frac{b}{a}ax\right)^\nu J_\mu \left[\left(\frac{b}{a}ax\right)^\nu\right] \nu \left(\frac{t}{a}\right)^{-1} (tx)^\nu J_\mu [(tx)^\nu] \\ \nu z^{-1} (zax)^\nu J_\mu [(zax)^\nu] adx \end{array} \right]$$

$$D_{\mu,\nu}(b/a, t/a, z) = \int_0^\infty (ax)^{-\mu\nu-\nu} \left[\begin{array}{l} \nu \left(\frac{b}{a}\right)^{-1} (bx)^\nu J_\mu [(bx)^\nu] \nu \left(\frac{t}{a}\right)^{-1} (tx)^\nu J_\mu [(tx)^\nu] \\ \nu z^{-1} (zax)^\nu J_\mu [(zax)^\nu] adx \end{array} \right]$$

Using the representation (8), the inner integral can be evaluated by means of the formula [3, p. 390 (4)]. Thus

$$\begin{aligned} & \int_0^\infty t^{-\tau} D_{\mu,\nu}(b/a, t/a, z) dt \\ &= \int_0^\infty t^{-\tau} \int_0^\infty (ax)^{-\mu\nu-\nu} \left[\begin{array}{l} \nu \left(\frac{b}{a}\right)^{-1} (bx)^\nu J_\mu [(bx)^\nu] \nu \left(\frac{t}{a}\right)^{-1} (tx)^\nu J_\mu [(tx)^\nu] \\ \nu z^{-1} (zax)^\nu J_\mu [(zax)^\nu] adx \end{array} \right] dt \\ &= \int_0^\infty a^{-\mu\nu-\nu+4} x^{-\mu\nu+2\nu} \nu^3 (zab)^{-1+\nu} J_\mu [(bx)^\nu] J_\mu [(zax)^\nu] \left(\int_0^\infty t^{-\tau-1+\nu} J_\mu [(tx)^\nu] dt \right) dx \\ &= \int_0^\infty a^{-\mu\nu-\nu+4} x^{-\mu\nu+2\nu} \nu^3 (zab)^{-1+\nu} J_\mu [(bx)^\nu] J_\mu [(zax)^\nu] \left(\frac{2^{\frac{-\tau}{\nu}} (x^\nu)^{\frac{\tau+\nu}{\nu}} x^{-2\nu} \Gamma\left(\frac{1}{2} \frac{\mu\nu - \tau + \nu}{\nu}\right)}{\nu \Gamma\left(\frac{1}{2} \frac{\mu\nu + \tau + \nu}{\nu}\right)} \right) dx \\ &= \frac{2^{\frac{-\tau}{\nu}} \Gamma\left(\frac{1}{2} \frac{\mu\nu - \tau + \nu}{\nu}\right) \nu^3 (zab)^{-1+\nu} a^{-\mu\nu+4-\nu} \int_0^\infty x^{\tau+\nu-\mu\nu} J_\mu [(bx)^\nu] J_\mu [(zax)^\nu] dx}{\nu \Gamma\left(\frac{1}{2} \frac{\mu\nu + \tau + \nu}{\nu}\right)}. \end{aligned}$$

Now evaluating the last integral by calculations, and substituting the value of the integral (25),

$$\begin{aligned} (H_{1,\nu,\mu,\psi}(t^{-\tau}))(b,a) &= a^{2\mu\nu+\nu} \left(\frac{2^{\frac{-\mu\nu+1}{\nu}} \Gamma\left(\frac{1}{2} \frac{\mu\nu-\tau+\nu}{\nu}\right) \Gamma\left(\frac{1}{2} \frac{\mu\nu+\tau+\nu+1}{\nu}\right)}{\Gamma\left(\frac{1}{2} \frac{\mu\nu+\tau+\nu}{\nu}\right) \Gamma\left(-\frac{1}{2} \frac{-\mu\nu+\tau-\nu+1}{\nu}\right) \Gamma(1+\mu)} \right) \times \\ &\quad \nu (zab)^{-1+\nu} (za)^{\mu\nu} b^{5\nu-\tau+1} a^{-\mu\nu+4-\nu-1/2} \times \\ &\quad \int_0^\infty {}_2F_1\left[\left[\frac{1}{2} \frac{\mu\nu+\tau+\nu+1}{\nu}, \frac{1}{2} \frac{-\mu\nu+\tau+\nu+1}{\nu}\right], [1+\mu], (za)^{2\nu} (b)^{-2\nu}\right] \psi(z) dz \end{aligned}$$

for $0 < \tau < \mu\nu + \nu$, $a, b > 0$.

Example 2. Assume that $f(t) = t^{-3} \sin(w_0 t)$; $w_0 > 0$.

Then,

$$\begin{aligned} &(H_{1,\nu,\mu,\psi}(t^{-3} \sin(w_0 t)))(b,a) \\ &= a^{-2\mu\nu-\nu} \int_0^\infty (t^{-3} \sin(w_0 t)) \psi\left(\frac{b}{a}, \frac{t}{a}\right) dt \\ &= a^{-2\mu\nu-\nu} \int_0^\infty \psi(z) \left(\int_0^\infty (t^{-3} \sin(w_0 t)) D_{\mu,\nu}(b/a, t/a, z) dt \right) dz. \end{aligned}$$

Using the representation (8), the inner integral can be evaluated by means of the formula [3, p. 405 (2)]. Thus

$$\begin{aligned} &\int_0^\infty t^{-3} \sin(w_0 t) D_{\mu,\nu}(b/a, t/a, z) dt \\ &= \int_0^\infty t^{-3} \sin(w_0 t) \int_0^\infty (ax)^{-\mu\nu-\nu} \left[\begin{aligned} &\nu \left(\frac{b}{a}\right)^{-1} (bx)^\nu J_\mu[(bx)^\nu] \nu \left(\frac{t}{a}\right)^{-1} (tx)^\nu J_\mu[(tx)^\nu] \\ &\nu z^{-1} (zax)^\nu J_\mu[(zax)^\nu] a dx \end{aligned} \right] dt \\ &= \int_0^\infty a^{-\mu\nu-\nu+4} x^{-\mu\nu+2\nu} \nu^3 (zab)^{-1} J_\mu[(bx)^\nu] J_\mu[(zax)^\nu] \left(\int_0^\infty \sin(w_0 t) t^{-4+\nu} J_\mu[(tx)^\nu] dt \right) dx \\ &= \int_0^\infty a^{-\mu\nu-\nu+4} x^{-\mu\nu+2\nu} \nu^3 (zab)^{-1+\nu} J_\mu[(bx)^\nu] J_\mu[(zax)^\nu] \left(\frac{2^{\frac{-4}{\nu}} x^{4+\nu} x^{-2\nu+\nu} w \Gamma\left(\frac{1}{2} \frac{\mu\nu-4+\nu}{\nu}\right)}{\nu \Gamma\left(\frac{1}{2} \frac{\mu\nu+4+\nu}{\nu}\right)} \right) dx \\ &= \left(\frac{2^{\frac{-4}{\nu}} x^4 w \Gamma\left(\frac{1}{2} \frac{\mu\nu-4+\nu}{\nu}\right)}{\nu \Gamma\left(\frac{1}{2} \frac{\mu\nu+4+\nu}{\nu}\right)} \right) \nu^3 (zab)^{-1+\nu} a^{-\mu\nu+4-\nu} \int_0^\infty x^{\nu-\mu\nu} J_\mu[(bx)^\nu] J_\mu[(zax)^\nu] dx. \end{aligned}$$

Now evaluating the last integral by calculations, and substituting the value of the integral (25),

$$\begin{aligned} \left(H_{1,\nu,\mu,\psi} \left(t^{-3} \sin(w_0 t) \right) \right) (b, a) &= a^{15/2} \left(\frac{2^{\frac{-5-6\nu+\mu\nu}{\nu}} x^4 w \Gamma \left(\frac{1}{2} \frac{\mu\nu-4+\nu}{\nu} \right) \Gamma \left(\frac{1}{2} \frac{\mu\nu-\tau+\nu}{\nu} \right) \Gamma \left(\frac{1}{2} \frac{\mu\nu+\nu+1}{\nu} \right)}{\nu \Gamma \left(\frac{1}{2} \frac{\mu\nu+4+\nu}{\nu} \right) \Gamma \left(\frac{1}{2} \frac{\mu\nu-\nu-1}{\nu} \right) \Gamma(1+\mu)} \right) \times \\ &\quad \nu^4 (za)^{\mu\nu-2+2\nu} b^{7\nu-\tau-1} \times \\ &\quad \int_0^\infty {}_2F_1 \left(\left[\frac{1}{2} \frac{\mu\nu+\nu+1}{\nu}, \frac{1}{2} \frac{-\mu\nu+\nu+1}{\nu} \right], [1+\mu], (za)^{2\nu} (b)^{-2\nu} \right) \psi(z) dz. \end{aligned}$$

4. Conclusion

The developed theory of the generalized Hankel-type wavelet transformation can be applied in signal processing, computer vision, seismology, turbulence, computer graphics, image processing, digital communication, approximation theory, numerical analysis and statistics. Generalized Hankel-type wavelet transformation can be used for complex information such as speech, images etc. to be decomposed into elementary forms.

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