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Navier –Stokes First Exact Transformation

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Abstract In this article the Navier – Stokes (NSE) exact transformations to the simpler equations is covered. This transformation is executed by classical methods of Mathematical Analysis. The solution of such equations is simpler than solution of the well known NSE. These new equations essentially facilitate the solutions of the Navier – Stokes Millennium Problem and different problems of numerous applications of Applied Mathematics in engineering.

Keywords Incompressible Fluid, Navier – Stokes Equations, Acceleration Divergence, Vector-Valued Function, Poisson Equation

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1. Introduction

1.1. General Data

The Navier–Stokes equations (NSE) in the case of incompressible flow are given by

$$\rho\vec{F} - \operatorname{grad} p + \mu\nabla^2\dot{\vec{u}} = \rho\ddot{\vec{u}} \tag{1}$$

with such equation is called a continuity equation (if $\rho = \text{Const}$) [1, p. 174]

$$\operatorname{div}\dot{\vec{u}} = \frac{\partial \dot{u}_x}{\partial x} + \frac{\partial \dot{u}_y}{\partial y} + \frac{\partial \dot{u}_z}{\partial z} = 0 \tag{2}$$

For incompressible flow if the continuity equation is equivalent to

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \dot{u}_x\frac{\partial\rho}{\partial x} + \dot{u}_y\frac{\partial\rho}{\partial y} + \dot{u}_z\frac{\partial\rho}{\partial z} = 0 \tag{2*}$$

Here, $\vec{F} = \vec{F}_1 + \vec{F}_2 + ...$ - vectors sum of a given, externally applied forces (e.g. gravity $\vec{F}_1$, magnetic $\vec{F}_2$ and other), p - pressure, $\dot{\vec{u}}$ - velocity vector, $\ddot{\vec{u}} = d\dot{\vec{u}}/dt$ - acceleration vector, ρ - density, μ - viscosity, ∇^2 - Laplace operator.

After taking an operator div of both sides (1) the NSE becomes [2, p. 74]

$$\rho \operatorname{div}\vec{F} - \operatorname{div}\operatorname{grad} p + \mu\nabla^2 \operatorname{div}\dot{\vec{u}} = \rho\operatorname{div}\ddot{\vec{u}},$$
$$(\mu, \rho = \text{Const}) \tag{1*}$$

Here, $\nabla^2 \operatorname{div}\dot{\vec{u}} = 0$, $\operatorname{div}\operatorname{grad} p = \nabla^2 p$ [3, p. 180]. Therefore equation (1*) can be written as

$$\nabla^2 p = -\rho\operatorname{div}(\ddot{\vec{u}} - \vec{F}). \tag{1**}$$

In the textbook [2] we can see equation (1**) if $\vec{F} = 0$ and velocity vector $\dot{\vec{u}}$ is a given function (therefore $\ddot{\vec{u}}$ is known). However, authors of textbook [2] could not prolong the NSE transformation.

1.2. General Aim

The aim of this paper is to prove that the NSE exact transformation to the simpler equations is possible. These new equations should facilitate the solution of Navier–Stokes existence and smoothness one of seven Millennium Prize Problems that were stated by the Clay Mathematics Institute. Also, new equations will simplify the solutions of many problems of Applied Mathematics in engineering.

2. Method of Transformation

First note that any vector on Euclidean space $\vec{u} = \vec{u}(x, y, z)$ can be represented as $\vec{u} = \vec{u}(\varsigma)$, $\varsigma = \varsigma(x, y, z)$. This representation is well known as a vector function of scalar argument (also called vector-valued function) [4, p. 514]. Therefore the velocity vector $\dot{\vec{u}} = \dot{\vec{u}}(x, y, z, t)$ for a fix time $t = \overline{t}$ can be represented as $\dot{\vec{u}} = \dot{\vec{u}}(\varsigma)$, $\varsigma = \varsigma(x, y, z)$. Then according to [4, p. 644] we obtain

$$\frac{\partial\dot{\vec{u}}}{\partial x_i} = \frac{\partial\dot{\vec{u}}}{\partial\varsigma}\frac{\partial\varsigma}{\partial x_i}, (x_i = x, y, z). \tag{3}$$

Formulas (3) can be written in component form

$$\frac{\partial\dot{\vec{u}}}{\partial x} = \frac{\partial\dot{\vec{u}}}{\partial\varsigma}\frac{\partial\varsigma}{\partial x}, \ \frac{\partial\dot{\vec{u}}}{\partial y} = \frac{\partial\dot{\vec{u}}}{\partial\varsigma}\frac{\partial\varsigma}{\partial y}, \ \frac{\partial\dot{\vec{u}}}{\partial z} = \frac{\partial\dot{\vec{u}}}{\partial\varsigma}\frac{\partial\varsigma}{\partial z}. \tag{3*}$$

Note that formulas (3*) also well known as chain rule. They can be written explicitly concerning of a common factor $\frac{\partial \dot{\vec{u}}}{\partial \varsigma}$. Therefore this common factor can be eliminated after usual transformations. As a result we have

$$\frac{\partial \dot{\vec{u}}}{\partial x_i} = \frac{\partial \dot{\vec{u}}}{\partial x_j} \frac{\partial \varsigma / \partial x_i}{\partial \varsigma / \partial x_j}. \tag{4}$$

In component form formulas (4) can be written so

$$\begin{gathered} \frac{\partial \dot{u}_x}{\partial x_i} = \frac{\partial \dot{u}_x}{\partial x_j} \frac{\partial \varsigma / \partial x_i}{\partial \varsigma / \partial x_j}, \quad \frac{\partial \dot{u}_y}{\partial x_i} = \frac{\partial \dot{u}_y}{\partial x_j} \frac{\partial \varsigma / \partial x_i}{\partial \varsigma / \partial x_j}, \\ \frac{\partial \dot{u}_z}{\partial x_i} = \frac{\partial \dot{u}_z}{\partial x_j} \frac{\partial \varsigma / \partial x_i}{\partial \varsigma / \partial x_j} \end{gathered}. \tag{5}$$

Note that relations (5) can be written explicitly concerning of another common factor $\frac{\partial \varsigma / \partial x_i}{\partial \varsigma / \partial x_j}$. Therefore this common factor can be eliminated. Thus

$$\begin{gathered} \frac{\partial \dot{u}_x}{\partial x_i} \frac{\partial \dot{u}_y}{\partial x_j} = \frac{\partial \dot{u}_x}{\partial x_j} \frac{\partial \dot{u}_y}{\partial x_i}, \quad \frac{\partial \dot{u}_x}{\partial x_i} \frac{\partial \dot{u}_z}{\partial x_j} = \frac{\partial \dot{u}_x}{\partial x_j} \frac{\partial \dot{u}_z}{\partial x_i}, \\ \frac{\partial \dot{u}_y}{\partial x_i} \frac{\partial \dot{u}_z}{\partial x_j} = \frac{\partial \dot{u}_y}{\partial x_j} \frac{\partial \dot{u}_z}{\partial x_i} \end{gathered}. \tag{5*}$$

Important note (extraordinary). Formulas (5*) means that we can not receive true solutions of any vector equations without these additional equalities. Therefore all solutions of the NSE and other vector PDE should satisfy these requirements.

3. NSE Exact Transformations

3.1. Transformations of Acceleration Divergence

The acceleration vector components $\ddot{u}_i$ can be written as follows from [1, p. 39]

$$\begin{aligned} \ddot{u}_x &= \frac{d\dot{u}_x}{dt} = \frac{\partial \dot{u}_x}{\partial t} + \dot{u}_x \frac{\partial \dot{u}_x}{\partial x} + \dot{u}_y \frac{\partial \dot{u}_x}{\partial y} + \dot{u}_z \frac{\partial \dot{u}_x}{\partial z}, \\ \ddot{u}_y &= \frac{d\dot{u}_y}{dt} = \frac{\partial \dot{u}_y}{\partial t} + \dot{u}_x \frac{\partial \dot{u}_y}{\partial x} + \dot{u}_y \frac{\partial \dot{u}_y}{\partial y} + \dot{u}_z \frac{\partial \dot{u}_y}{\partial z}, \\ \ddot{u}_z &= \frac{d\dot{u}_z}{dt} = \frac{\partial \dot{u}_z}{\partial t} + \dot{u}_x \frac{\partial \dot{u}_z}{\partial x} + \dot{u}_y \frac{\partial \dot{u}_z}{\partial y} + \dot{u}_z \frac{\partial \dot{u}_z}{\partial z}. \end{aligned} \tag{6}$$

After taking an operator div we obtain

$$\begin{aligned} \operatorname{div}\ddot{\vec{u}} &= \frac{\partial \ddot{u}_x}{\partial x} + \frac{\partial \ddot{u}_y}{\partial y} + \frac{\partial \ddot{u}_z}{\partial z} = \frac{\partial}{\partial t}\operatorname{div}\dot{\vec{u}} + \dot{u}_x \frac{\partial}{\partial x}\operatorname{div}\dot{\vec{u}} + \\ &\dot{u}_y \frac{\partial}{\partial y}\operatorname{div}\dot{\vec{u}} + \dot{u}_z \frac{\partial}{\partial z}\operatorname{div}\dot{\vec{u}} + \left(\frac{\partial \dot{u}_x}{\partial x}\right)^2 + \left(\frac{\partial \dot{u}_y}{\partial y}\right)^2 + \left(\frac{\partial \dot{u}_z}{\partial z}\right)^2 \\ &+2\left(\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x} + \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y} + \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}\right). \end{aligned}$$

This formula can be written as

$$\operatorname{div}\ddot{\vec{u}} = \frac{d}{dt}\operatorname{div}\dot{\vec{u}} + (\operatorname{div}\dot{\vec{u}})^2 \tag{7}$$

if and only if this equality is true

$$\begin{aligned} &\left(\frac{\partial \dot{u}_x}{\partial x}\right)^2 + \left(\frac{\partial \dot{u}_y}{\partial y}\right)^2 + \left(\frac{\partial \dot{u}_z}{\partial z}\right)^2 + \\ &2\left(\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x} + \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y} + \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}\right) = (\operatorname{div}\dot{\vec{u}})^2 \end{aligned} \tag{7*}$$

The equality (7*) requires such equality

$$\begin{aligned} &\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x} + \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y} + \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x} = \\ &\frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_y}{\partial y} + \frac{\partial \dot{u}_y}{\partial y}\frac{\partial \dot{u}_z}{\partial z} + \frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_z}{\partial z} \end{aligned} \tag{7**}$$

Let's substitute $x_i = x, y, z, \ (x_j \neq x_i)$ into (5*). Finally, we obtain

$$\begin{gathered} \frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_y}{\partial y} = \frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x}, \frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_z}{\partial z} = \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}, \\ \frac{\partial \dot{u}_y}{\partial y}\frac{\partial \dot{u}_z}{\partial z} = \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y} \end{gathered} \tag{8}$$

Note that the presence of (8) are necessary conditions for equalities (7*) and (7**). In that case we can receive equations (7). But below (in 3.2) we will show that equations (7) can be obtained by another way.

3.2. NSE Transformation

From (7) we can see that $\operatorname{div}\ddot{\vec{u}} = 0$ if $\operatorname{div}\dot{\vec{u}} = 0$. In that case formula (1**) can be conversed to a Poisson equation

$$\nabla^2 p = \rho \operatorname{div}\vec{F}.$$

Therefore $\nabla^2 p = 0$ if $\operatorname{div}\vec{F} = 0$. Now we can eliminate the pressure p by taking operator ∇^2 of both sides of equations (1). After these transformations the NSE can be written as (if $\operatorname{div}\vec{F} = 0$)

$$\nabla^2\left(\mu\nabla^2\dot{u}_i - \rho(\ddot{u}_i - F_i)\right) = 0. \tag{9}$$

After notation

$$\dot{u}_i = \int \ddot{u}_i dt \tag{10}$$

we obtain the following general equations for incompressible fluids if $\operatorname{div}\vec{F} = 0, \ \rho = \text{Const}$

$$\begin{aligned} &\nabla^2 p = 0, \quad \nabla^2\left(\nu\nabla^2\int \ddot{u}_i dt - (\ddot{u}_i - F_i)\right) = 0, \\ &\nu = \frac{\mu}{\rho}, \quad (i = 1,2,3). \end{aligned} \tag{11}$$

3.3. Proof of Equation (7) from the Point of View of Continuum Mechanics

The authors of textbook [2] could not prolong NSE transformation because they did not consider this analogy which very well known in continuum mechanics [1, p. 107]

$$\varepsilon_o = \frac{1}{\delta V} d(\delta V) = \operatorname{div} \vec{u},\ \dot{\varepsilon}_o = \frac{1}{\delta V}\frac{d(\delta V)}{dt} = \operatorname{div} \dot{\vec{u}}. \qquad (12)$$

Here, ε_o - volume deformation, $\vec{u}$ - infinitesimal displacement vector ($\vec{u}$ - any displacement vector), $\dot{\varepsilon}_o$ -velocity of volume deformation.

By above analogy, the acceleration divergence $\operatorname{div}\ddot{\vec{u}}$ can be written as ($\ddot{\varepsilon}_o$ - acceleration of volume deformation)

$$\ddot{\varepsilon}_o = \frac{1}{\delta V}\frac{d^2(\delta V)}{dt^2} = \operatorname{div}\ddot{\vec{u}} \qquad (12^*)$$

As we can see this transformation of (12*) is exact:

$$\ddot{\varepsilon}_o = \frac{1}{\delta V}\frac{d^2(\delta V)}{dt^2} = \frac{1}{\delta V}\frac{d}{dt}\left(\frac{\delta V}{\delta V}\frac{d(\delta V)}{dt}\right) = \frac{1}{\delta V}\frac{d}{dt}\left(\delta V \operatorname{div}\dot{\vec{u}}\right) =$$

$$\frac{d}{dt}\operatorname{div}\dot{\vec{u}} + \operatorname{div}\dot{\vec{u}}\frac{d(\delta V)}{\delta V dt} = \frac{d}{dt}\operatorname{div}\dot{\vec{u}} + (\operatorname{div}\dot{\vec{u}})^2.$$

Hence according to (12*) we obtain

$$\operatorname{div}\ddot{\vec{u}} = \frac{d}{dt}\operatorname{div}\dot{\vec{u}} + (\operatorname{div}\dot{\vec{u}})^2 .$$

As we can see it is equation (7), which we obtain above by another method. This result improves the visibility of equations (3) - (5) which were used for exact proof of equation (7).

4. Discussion of Main Results

After comparison the above results we can see that transforming NSE system (11) are simpler than system (1),(2). Each of four equations (11) include only one of four unknown functions – pressure p and three components of the velocity vector $\ddot{u}_x, \ddot{u}_y, \ddot{u}_z$. These new equations significantly facilitate the solutions of the Navier – Stokes Millennium Problem and different problems of Applied Mathematics in engineering. For the partial solutions of (11) it is possible to use the traditional boundary conditions: the acceleration vector $\ddot{\vec{u}}$, as well as the velocity vector $\dot{\vec{u}}$, is a zero on the immobile boundaries. Also, an examples of exact solutions of Laplace's equation $\nabla^2 p = 0$ are well known.

5. Conclusion

In this article the NSE exact transformation to the significantly simpler equations is covered. This transformation is executed by classical methods of the Mathematical Analysis. These new equations essentially facilitate the solution of the Navier–Stokes existence and smoothness one of the Millennium Prize Problems . Equation (1**) is obtained by Landau and Lifshits if $\vec{F} = 0$. These results are covered very well in the textbook [2, p. 74]. However Landau and Lifshits could not prove that the right hand side of equation (1**) is a zero. The above results contain such proof. However any exact solutions of equations (11) can appear wrong for not coordinated or not smoothed initial and boundary conditions (about this problem for a wave equation read in [5, p. 63-83]).

In this paper only first NSE exact transformation is covered. Second exact transformation will be an object of another paper.

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