The Partial Differential Problem

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Abstract In calculus and engineering mathematics courses, the evaluation of the partial derivatives of multivariable functions is important. This paper takes the mathematical software Maple as the auxiliary tool to study the partial differential problem of two types of multivariable functions. We can obtain the infinite series forms of any order partial derivatives of these two types of multivariable functions by using differentiation term by term theorem, and hence greatly reduce the difficulty of calculating their higher order partial derivative values. On the other hand, we provide two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

Keywords Partial Derivatives, Infinite Series Forms, Differentiation Term By Term Theorem, Maple

1. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests. Inquiring through an online support system provided by

Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. As for the instructions and operations of Maple, we can refer to [1-7].

In calculus and engineering mathematics curricula, the evaluation and numerical calculation of the partial derivatives of multivariable functions are important. For example, Laplace equation, wave equation, as well as some other important physical equations are involved the partial derivatives. On the other hand, evaluating the m -th order partial derivative value of a multivariable function at some point, in general, needs to go through two procedures: firstly determining the m -th order partial derivative of this function, and then taking the point into this m -th order partial derivative. These two procedures will make us face with increasingly complex calculations when calculating higher order partial derivative values (i.e. m is large), and hence to obtain the answers by manual calculations is not easy. In this paper, we study the partial differential problem of the following two types of n -variables functions

$$f(x_1, x_2, \dots, x_n) = \prod_{k=1}^n x_k^{a_k} \cdot \tan^{-1} \left(\prod_{k=1}^n x_k^{b_k} \right) \quad (1)$$

$$g(x_1, x_2, \dots, x_n) = \prod_{k=1}^n x_k^{a_k} \cdot \cot^{-1} \left(\prod_{k=1}^n x_k^{b_k} \right) \quad (2)$$

Where n is a positive integer, a_k, b_k are real numbers for all $k = 1, \dots, n$. We can obtain the infinite series forms of any order partial derivatives of these two types of n -variables functions by using differentiation term by term theorem; these are the major results of this study (i.e., Theorems 1 and 2), and hence greatly reduce the difficulty of calculating their higher order partial derivative values. For the study of related partial differential problems can refer to [8-19]. In addition, we propose two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding

problem-solving methods.

2. Main Results

Firstly, we introduce some notations and formulas used in this paper.

2.1. Notations

2.1.1. $\prod_{k=1}^n a_k = a_1 \times a_2 \times \cdots \times a_n$, where n is a positive integer, a_k are real numbers for all $k = 1, \dots, n$.

2.1.2. Suppose r is any real number, m is any positive integer. Define $(r)_m = r(r-1)\cdots(r-m+1)$, and $(r)_0 = 1$.

2.1.3. Suppose n is a positive integer, j_k are non-negative integers for all $k = 1, \dots, n$. For the n -variables function $f(x_1, x_2, \dots, x_n)$, its j_k -times partial derivative with respect to x_k for all $k = 1, \dots, n$, forms a $j_1 + j_2 + \cdots + j_n$ -th order partial derivative, and denoted by $\frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n)$.

2.2. Formulas ([20])

2.2.1. Suppose y is a Real Number, the Inverse Tangent Function

$$\tan^{-1} y = \begin{cases} \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} y^{2p+1} & \text{if } |y| < 1 \\ \frac{\pi}{2} - \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} y^{-(2p+1)} & \text{if } y > 1 \\ -\frac{\pi}{2} - \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} y^{-(2p+1)} & \text{if } y < -1 \end{cases} \quad (3)$$

2.2.2. Suppose y is a Real Number, the Inverse Cotangent Function

$$\cot^{-1} y = \begin{cases} \frac{\pi}{2} - \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} y^{2p+1} & \text{if } |y| < 1 \\ \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} y^{-(2p+1)} & \text{if } y > 1 \\ \pi + \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} y^{-(2p+1)} & \text{if } y < -1 \end{cases} \quad (6)$$

Next, we introduce an important theorem used in this study.

2.3. Differentiation Term by Term Theorem([21])

For all non-negative integer k , if the functions $g_k : (a, b) \rightarrow R$ satisfy the following three conditions : (i) there exists a point $x_0 \in (a, b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a, b) , (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a, b) . Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a, b) . Moreover, its derivative $\frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

The following is the first result in this study, we determine the infinite series forms of any order partial derivatives of the multivariable function (1).

2.4. Theorem 1 Suppose n is a Positive Integer, a_k, b_k Are Real Numbers, and j_k are Non-Negative Integers For All $k = 1, \dots, n$. If the n -Variables Function

$$f(x_1, x_2, \dots, x_n) = \prod_{k=1}^n x_k^{a_k} \cdot \tan^{-1} \left(\prod_{k=1}^n x_k^{b_k} \right)$$

satisfies $x_k^{a_k}, x_k^{b_k}$ exist, $x_k \neq 0$ for all $k = 1, \dots, n$, and $\left| \prod_{k=1}^n x_k^{b_k} \right| \neq 1$.

Case (A). If $\left| \prod_{k=1}^n x_k^{b_k} \right| < 1$, then the $j_1 + j_2 + \cdots + j_n$ -th order partial derivative of $f(x_1, x_2, \dots, x_n)$,

$$\begin{aligned} & \frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n) \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} \prod_{k=1}^n ((2p+1)b_k + a_k)_{j_k} \prod_{k=1}^n x_k^{(2p+1)b_k + a_k - j_k} \end{aligned} \quad (9)$$

Case (B). If $\prod_{k=1}^n x_k^{b_k} > 1$, then

$$\begin{aligned} & \frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n) \\ &= \frac{\pi}{2} \cdot \prod_{k=1}^n (a_k)_{j_k} \prod_{k=1}^n x_k^{a_k - j_k} \\ &- \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} \prod_{k=1}^n (-(2p+1)b_k + a_k)_{j_k} \cdot \prod_{k=1}^n x_k^{-(2p+1)b_k + a_k - j_k} \end{aligned} \quad (10)$$

Case (C). If $\prod_{k=1}^n x_k^{b_k} < -1$, then

$$\begin{aligned}
& \frac{\partial^{j_1+j_2+\dots+j_n} f}{\partial x_n^{j_n} \dots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n) \\
&= -\frac{\pi}{2} \cdot \prod_{k=1}^n (a_k)_{j_k} \prod_{k=1}^n x_k^{a_k-j_k} \\
&= -\frac{\pi}{2} \cdot \prod_{k=1}^n (a_k)_{j_k} \prod_{k=1}^n x_k^{a_k-j_k} \\
&= -\sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n ((-2p+1)b_k + a_k)_{j_k} \cdot \prod_{k=1}^n x_k^{-(2p+1)b_k+a_k-j_k} \quad (11)
\end{aligned}$$

2.4.1. Proof Case (A). If $\left| \prod_{k=1}^n x_k^{b_k} \right| < 1$. Because

$$\begin{aligned}
& f(x_1, x_2, \dots, x_n) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \tan^{-1} \left(\prod_{k=1}^n x_k^{b_k} \right) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \left(\prod_{k=1}^n x_k^{b_k} \right)^{2p+1} \quad (\text{Using (3)}) \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n x_k^{(2p+1)b_k+a_k} \quad (12)
\end{aligned}$$

By differentiation term by term theorem, differentiating j_k -times with respect to x_k ($k=1, \dots, n$) on both sides of (12), we obtain the $j_1 + j_2 + \dots + j_n$ -th order partial derivative of $f(x_1, x_2, \dots, x_n)$,

$$\begin{aligned}
& \frac{\partial^{j_1+j_2+\dots+j_n} f}{\partial x_n^{j_n} \dots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n) \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n ((2p+1)b_k + a_k)_{j_k} \prod_{k=1}^n x_k^{(2p+1)b_k+a_k-j_k}
\end{aligned}$$

Case (B). If $\prod_{k=1}^n x_k^{b_k} > 1$. Because

$$\begin{aligned}
& f(x_1, x_2, \dots, x_n) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \tan^{-1} \left(\prod_{k=1}^n x_k^{b_k} \right) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \left[\frac{\pi}{2} - \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \left(\prod_{k=1}^n x_k^{b_k} \right)^{-(2p+1)} \right] \quad (\text{By (4)}) \\
&= \frac{\pi}{2} \cdot \prod_{k=1}^n x_k^{a_k} - \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n x_k^{-(2p+1)b_k+a_k} \quad (13)
\end{aligned}$$

Using differentiation term by term theorem, differentiating j_k -times with respect to x_k ($k=1, \dots, n$) on both sides of (13), we have

$$\frac{\partial^{j_1+j_2+\dots+j_n} f}{\partial x_n^{j_n} \dots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n)$$

Case (C). If $\prod_{k=1}^n x_k^{b_k} < -1$. Because

$$\begin{aligned}
& f(x_1, x_2, \dots, x_n) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \tan^{-1} \left(\prod_{k=1}^n x_k^{b_k} \right) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \left[-\frac{\pi}{2} - \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \left(\prod_{k=1}^n x_k^{b_k} \right)^{-(2p+1)} \right] \\
& \quad (\text{Using (5)}) \\
&= -\frac{\pi}{2} \cdot \prod_{k=1}^n x_k^{a_k} - \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n x_k^{-(2p+1)b_k+a_k} \quad (14)
\end{aligned}$$

Using differentiation term by term theorem, differentiating j_k -times with respect to x_k ($k=1, \dots, n$) on both sides of (14), we obtain

$$\begin{aligned}
& \frac{\partial^{j_1+j_2+\dots+j_n} f}{\partial x_n^{j_n} \dots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n) \\
&= -\frac{\pi}{2} \cdot \prod_{k=1}^n (a_k)_{j_k} \prod_{k=1}^n x_k^{a_k-j_k} \\
&= -\sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n ((-2p+1)b_k + a_k)_{j_k} \cdot \prod_{k=1}^n x_k^{-(2p+1)b_k+a_k-j_k}
\end{aligned}$$

Next, we determine the infinite series forms of any order partial derivatives of the multivariable function (2).

2.5. Theorem 2 Let the Assumptions be the Same as Theorem 1. Suppose the n -Variables Function

$$g(x_1, x_2, \dots, x_n) = \prod_{k=1}^n x_k^{a_k} \cdot \cot^{-1} \left(\prod_{k=1}^n x_k^{b_k} \right)$$

satisfies $x_k^{a_k}, x_k^{b_k}$ exist, $x_k \neq 0$ for all $k=1, \dots, n$, and

$$\left| \prod_{k=1}^n x_k^{b_k} \right| \neq 1$$

Case (A). If $\left| \prod_{k=1}^n x_k^{b_k} \right| < 1$, then the $j_1 + j_2 + \dots + j_n$ -th order partial derivative of $g(x_1, x_2, \dots, x_n)$,

$$\frac{\partial^{j_1+j_2+\dots+j_n} g}{\partial x_n^{j_n} \dots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n)$$

$$\begin{aligned}
&= \frac{\pi}{2} \cdot \prod_{k=1}^n (a_k)_{j_k} \prod_{k=1}^n x_k^{a_k - j_k} \\
&- \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n ((2p+1)b_k + a_k)_{j_k} \cdot \prod_{k=1}^n x_k^{(2p+1)b_k + a_k - j_k}
\end{aligned} \tag{15}$$

Case (B). If $\prod_{k=1}^n x_k^{b_k} > 1$, then

$$\begin{aligned}
&\frac{\partial^{j_1+j_2+\dots+j_n} g}{\partial x_n^{j_n} \dots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n) \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n (-(2p+1)b_k + a_k)_{j_k} \cdot \prod_{k=1}^n x_k^{-(2p+1)b_k + a_k - j_k}
\end{aligned} \tag{16}$$

Case (C). If $\prod_{k=1}^n x_k^{b_k} < -1$, then

$$\begin{aligned}
&\frac{\partial^{j_1+j_2+\dots+j_n} g}{\partial x_n^{j_n} \dots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n) \\
&= \pi \cdot \prod_{k=1}^n (a_k)_{j_k} \prod_{k=1}^n x_k^{a_k - j_k} \\
&+ \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n (-(2p+1)b_k + a_k)_{j_k} \cdot \prod_{k=1}^n x_k^{-(2p+1)b_k + a_k - j_k}
\end{aligned} \tag{17}$$

2.5.1. Proof Case (A). If $\left| \prod_{k=1}^n x_k^{b_k} \right| < 1$. Because

$$\begin{aligned}
&g(x_1, x_2, \dots, x_n) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \cot^{-1} \left(\prod_{k=1}^n x_k^{b_k} \right) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \left[\frac{\pi}{2} - \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \left(\prod_{k=1}^n x_k^{b_k} \right)^{2p+1} \right] \text{ (Using (6))} \\
&= \frac{\pi}{2} \cdot \prod_{k=1}^n x_k^{a_k} - \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n x_k^{(2p+1)b_k + a_k}
\end{aligned} \tag{18}$$

Using differentiation term by term theorem, differentiating j_k -times with respect to x_k ($k=1, \dots, n$) on both sides of (18), we obtain

$$\begin{aligned}
&\frac{\partial^{j_1+j_2+\dots+j_n} g}{\partial x_n^{j_n} \dots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n) \\
&= \frac{\pi}{2} \cdot \prod_{k=1}^n (a_k)_{j_k} \prod_{k=1}^n x_k^{a_k - j_k}
\end{aligned}$$

$$- \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n ((2p+1)b_k + a_k)_{j_k} \cdot \prod_{k=1}^n x_k^{(2p+1)b_k + a_k - j_k} \tag{19}$$

Case (B). If $\prod_{k=1}^n x_k^{b_k} > 1$. Because

$$\begin{aligned}
&g(x_1, x_2, \dots, x_n) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \cot^{-1} \left(\prod_{k=1}^n x_k^{b_k} \right) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \left(\prod_{k=1}^n x_k^{b_k} \right)^{-(2p+1)} \text{ (By (7))} \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n x_k^{-(2p+1)b_k + a_k}
\end{aligned} \tag{20}$$

By differentiation term by term theorem, differentiating j_k -times with respect to x_k ($k=1, \dots, n$) on both sides of (20), we have

$$\begin{aligned}
&\frac{\partial^{j_1+j_2+\dots+j_n} g}{\partial x_n^{j_n} \dots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n) \\
&= \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n (-(2p+1)b_k + a_k)_{j_k} \cdot \prod_{k=1}^n x_k^{-(2p+1)b_k + a_k - j_k}
\end{aligned}$$

Case (C). If $\prod_{k=1}^n x_k^{b_k} < -1$. Because

$$\begin{aligned}
&g(x_1, x_2, \dots, x_n) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \cot^{-1} \left(\prod_{k=1}^n x_k^{b_k} \right) \\
&= \prod_{k=1}^n x_k^{a_k} \cdot \left[\pi + \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \left(\prod_{k=1}^n x_k^{b_k} \right)^{-(2p+1)} \right] \text{ (Using (8))} \\
&= \pi \cdot \prod_{k=1}^n x_k^{a_k} + \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{p+1}} \prod_{k=1}^n x_k^{-(2p+1)b_k + a_k}
\end{aligned} \tag{21}$$

Also, by differentiation term by term theorem, differentiating j_k -times with respect to x_k ($k=1, \dots, n$) on both sides of (21), we have

$$\begin{aligned}
&\frac{\partial^{j_1+j_2+\dots+j_n} g}{\partial x_n^{j_n} \dots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \dots, x_n) \\
&= \pi \cdot \prod_{k=1}^n (a_k)_{j_k} \prod_{k=1}^n x_k^{a_k - j_k}
\end{aligned}$$

$$+ \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} \prod_{k=1}^n (-(2p+1)b_k + a_k)_{j_k} \cdot \prod_{k=1}^n x_k^{-(2p+1)b_k + a_k - j_k}$$

$$f := (x_1, x_2) \rightarrow \frac{x_1^3 \arctan\left(\frac{x_2^{3/2}}{x_1^7}\right)}{x_2^{12/5}}$$

3. Examples

In the following, for the partial differential problem of the two types of multivariable functions in this study, we provide two examples and use Theorems 1 and 2 to determine the infinite series forms of any order partial derivatives and some higher order partial derivative values of these functions. On the other hand, we employ Maple to calculate the approximations of these higher order partial derivative values and their solutions for verifying our answers.

3.1. Example 1 If The Domain of the Two-Variables Function

$$f(x_1, x_2) = x_1^3 x_2^{-12/5} \tan^{-1}(x_1^{-7} x_2^{3/2}) \quad (22)$$

Is

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \neq 0, x_2 > 0 \right\}$$

Case (1). If $|x_1^{-7} x_2^{3/2}| < 1$. By Case (A) of Theorem 1, we obtain any $j_1 + j_2$ -th order partial derivative of $f(x_1, x_2)$,

$$\begin{aligned} & \frac{\partial^{j_1+j_2} f}{\partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2) \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} (-14p-4)_{j_1} \left(3p - \frac{9}{10}\right)_{j_2} \cdot \\ & \quad x_1^{-14p-4-j_1} \cdot x_2^{3p-\frac{9}{10}-j_2} \end{aligned} \quad (23)$$

For all $x_1 \neq 0, x_2 > 0, |x_1^{-7} x_2^{3/2}| < 1$. Hence, we can evaluate the 15-th order partial derivative value of $f(x_1, x_2)$ at $\left(2, \frac{11}{3}\right)$,

$$\begin{aligned} & \frac{\partial^{15} f}{\partial x_2^6 \partial x_1^9} \left(2, \frac{11}{3}\right) \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} (-14p-4)_9 \left(3p - \frac{9}{10}\right)_6 \cdot \\ & \quad 2^{-14p-13} \cdot \left(\frac{11}{3}\right)^{3p-\frac{69}{10}} \end{aligned} \quad (24)$$

We use Maple to verify the correctness of (24) as follows:

$$>f:=(x1,x2)->x1^3*x2^(-12/5)*arctan(x1^(-7)*x2^(3/2));$$

$$>\text{evalf}(D[1\$9,2\$6](f)(2,11/3),18);$$

$$1077.05802477880989$$

$$>\text{evalf}(\text{sum}((-1)^p/(2*p+1)*\text{product}(-14*p-4-j,j=0..8)*\text{product}(3*p-9/10-r,r=0..5)*2^(-14*p-13)*(11/3)^(3*p-69/10),p=\text{infinity}),18);$$

$$1077.05802477880989$$

Case (2). If $x_1^{-7} x_2^{3/2} > 1$. By Case (B) of Theorem 1, we obtain any $j_1 + j_2$ -th order partial derivative of $f(x_1, x_2)$,

$$\begin{aligned} & \frac{\partial^{j_1+j_2} f}{\partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2) \\ &= \frac{\pi}{2} \cdot (3)_{j_1} \left(-\frac{12}{5}\right)_{j_2} \cdot x_1^{3-j_1} \cdot x_2^{-\frac{12}{5}-j_2} \\ & \quad - \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} (14p+10)_{j_1} \left(-3p - \frac{39}{10}\right)_{j_2} \cdot \\ & \quad x_1^{14p+10-j_1} \cdot x_2^{-3p-\frac{39}{10}-j_2} \end{aligned} \quad (25)$$

For all $x_1 \neq 0, x_2 > 0, x_1^{-7} x_2^{3/2} > 1$. Thus, we can determine the 17-th order partial derivative value of $f(x_1, x_2)$ at $\left(\frac{1}{2}, \frac{3}{5}\right)$,

$$\begin{aligned} & \frac{\partial^{17} f}{\partial x_2^{10} \partial x_1^7} \left(\frac{1}{2}, \frac{3}{5}\right) \\ &= \frac{\pi}{2} \cdot (3)_7 \left(-\frac{12}{5}\right)_{10} \cdot \left(\frac{1}{2}\right)^{-4} \cdot \left(\frac{3}{5}\right)^{-\frac{62}{5}} \\ & \quad - \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} (14p+10)_7 \left(-3p - \frac{39}{10}\right)_{10} \cdot \\ & \quad \left(\frac{1}{2}\right)^{14p+3} \cdot \left(\frac{3}{5}\right)^{-3p-\frac{139}{10}} \end{aligned} \quad (26)$$

Using Maple to verify the correctness of (26) as follows:

$$>\text{evalf}(D[1\$7,2\$10](f)(1/2,3/5),28);$$

$$5.286881883955459358000444256 \cdot 10^{17}$$

$$>\text{evalf}(\text{Pi}/2*\text{product}(3-t,t=0..6)*\text{product}(-12/5-w,w=0..9)*(1/2)^(-4)*(3/5)^(-62/5)-\text{sum}((-1)^p/(2*p+1)*\text{product}(14*p+10-j,j=0..6)*\text{product}(-3*p-39/10-r,r=0..9)*(1/2)^(14*p+3)*(3/5)^(-3*p-139/10),p=0..infinity),28);$$

$$5.286881883955459358000444256 \cdot 10^{17}$$

Case (3). If $x_1^{-7}x_2^{3/2} < -1$. By Case (C) of Theorem 1, we can determine any $j_1 + j_2$ -th order partial derivative of $f(x_1, x_2)$,

$$\begin{aligned} & \frac{\partial^{j_1+j_2} f}{\partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2) \\ &= -\frac{\pi}{2} \cdot (3)_{j_1} \left(-\frac{12}{5}\right)_{j_2} \cdot x_1^{3-j_1} \cdot x_2^{\frac{12}{5}-j_2} \\ & - \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} (14p+10)_{j_1} \left(-3p-\frac{39}{10}\right)_{j_2} \cdot \\ & x_1^{14p+10-j_1} \cdot x_2^{-3p-\frac{39}{10}-j_2} \end{aligned} \quad (27)$$

For all $x_1 \neq 0, x_2 > 0$, $x_1^{-7}x_2^{3/2} < -1$. Thus, we can evaluate the 13-th order partial derivative value of $f(x_1, x_2)$

$$\begin{aligned} & \text{at } \left(-\frac{1}{3}, \frac{5}{4}\right), \\ & \frac{\partial^{13} f}{\partial x_2^8 \partial x_1^5} \left(-\frac{1}{3}, \frac{5}{4}\right) \\ &= -\frac{\pi}{2} \cdot (3)_5 \left(-\frac{12}{5}\right)_8 \cdot \left(-\frac{1}{3}\right)^{-2} \cdot \left(\frac{5}{4}\right)^{-\frac{52}{5}} \\ & - \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} (14p+10)_5 \left(-3p-\frac{39}{10}\right)_8 \cdot \\ & \left(-\frac{1}{3}\right)^{14p+5} \cdot \left(\frac{5}{4}\right)^{-3p-\frac{119}{10}} \end{aligned} \quad (28)$$

Also, we use Maple to verify the correctness of (28).

>evalf(D[1\$5,2\$8](f)(-1/3,5/4),14);

$$5.1610429136955 \cdot 10^7$$

>evalf(-Pi/2*product(3-t,t=0..4)*product(-12/5-w,w=0..7)*(-1/3)^(-2)*(5/4)^(-52/5)-sum((-1)^p/(2*p+1)*product(14*p+10-j,j=0..4)*product(-3*p-39/10-r,r=0..7)*(-1/3)^(14*p+5)*(5/4)^(-3*p-119/10),p=0..infinity),14);

$$5.1610429136955 \cdot 10^7$$

3.2. Example 2 Let the Domain of the Three-Variables Function

$$g(x_1, x_2, x_3) = x_1^{-3/4} x_2^6 x_3^{-7/2} \cot^{-1}(x_1^{11/3} x_2^{-5} x_3^{8/9}) \quad (29)$$

$$\text{be } \{x_1, x_2, x_3\} \in R^3 \mid x_1 > 0, x_2 \neq 0, x_3 > 0\}.$$

Case (1). If $|x_1^{11/3} x_2^{-5} x_3^{8/9}| < 1$. By Case (A) of

Theorem 2, we obtain any $j_1 + j_2 + j_3$ -th order partial derivative of $g(x_1, x_2, x_3)$,

$$\begin{aligned} & \frac{\partial^{j_1+j_2+j_3} g}{\partial x_3^{j_3} \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, x_3) \\ &= \frac{\pi}{2} \cdot \left(-\frac{3}{4}\right)_{j_1} (6)_{j_2} \left(-\frac{7}{2}\right)_{j_3} \cdot x_1^{-\frac{3}{4}-j_1} \cdot x_2^{6-j_2} \cdot x_3^{-\frac{7}{2}-j_3} \\ & - \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} \left(\frac{22}{3}p + \frac{35}{12}\right)_{j_1} (-10p+1)_{j_2} \left(\frac{16}{9}p - \frac{47}{18}\right)_{j_3} \times \\ & x_1^{\frac{22}{3}p + \frac{35}{12}-j_1} \cdot x_2^{-10p+1-j_2} \cdot x_3^{\frac{16}{9}p - \frac{47}{18}-j_3} \end{aligned} \quad (30)$$

For all $x_1 > 0, x_2 \neq 0, x_3 > 0$, $|x_1^{11/3} x_2^{-5} x_3^{8/9}| < 1$. Hence, we can evaluate the 16-th order partial derivative value of $g(x_1, x_2, x_3)$ at $\left(\frac{3}{4}, \frac{5}{2}, \frac{7}{9}\right)$,

$$\begin{aligned} & \frac{\partial^{16} g}{\partial x_3^7 \partial x_2^5 \partial x_1^4} \left(\frac{3}{4}, \frac{5}{2}, \frac{7}{9}\right) \\ &= \frac{\pi}{2} \cdot \left(-\frac{3}{4}\right)_4 (6)_5 \left(-\frac{7}{2}\right)_7 \cdot \left(\frac{3}{4}\right)^{-\frac{19}{4}} \cdot \left(\frac{5}{2}\right)^1 \cdot \left(\frac{7}{9}\right)^{-\frac{21}{2}} \\ & - \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} \left(\frac{22}{3}p + \frac{35}{12}\right)_4 (-10p+1)_5 \left(\frac{16}{9}p - \frac{47}{18}\right)_7 \cdot \\ & \left(\frac{3}{4}\right)^{\frac{22}{3}p - \frac{13}{12}} \cdot \left(\frac{5}{2}\right)^{-10p-4} \cdot \left(\frac{7}{9}\right)^{\frac{16}{9}p - \frac{173}{18}} \end{aligned} \quad (31)$$

Verifying the correctness of (31) as follows:

>g:=(x1,x2,x3)->x1^(-3/4)*x2^6*x3^(-7/2)*arccot(x1^(11/3)*x2^(-5)*x3^(8/9));

$$g := (x1, x2, x3) \rightarrow \frac{x2^6 \arccot\left(\frac{x1^{11/3} x3^{8/9}}{x2^5}\right)}{x1^{3/4} x3^{7/2}}$$

>evalf(D[1\$4,2\$5,3\$7](g)(3/4,5/2,7/9),18);

$$-7.16292131888003098 \cdot 10^{11}$$

>evalf(Pi/2*product(-3/4-t,t=0..3)*product(6-w,w=0..4)*product(-7/2-s,s=0..6)*(3/4)^(-19/4)*(5/2)^1*(7/9)^(-21/2)-sum((-1)^p/(2*p+1)*product(22*p/3+35/12-j,j=0..3)*product(-10*p+1-r,r=0..4)*product(16*p/9-47/18-u,u=0..6)*(3/4)^(22*p/3-13/12)*(5/2)^(-10*p-4)*(7/9)^(16*p/9-173/18),p=0..infinity),18);

$$-7.16292131888003094 \cdot 10^{11}$$

Case (2). If $|x_1^{11/3} x_2^{-5} x_3^{8/9}| > 1$. Using Case (B) of

Theorem 2, we obtain any $j_1 + j_2 + j_3$ -th order partial derivative of $g(x_1, x_2, x_3)$,

$$\begin{aligned} & \frac{\partial^{j_1+j_2+j_3} g}{\partial x_3^{j_3} \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, x_3) \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} \left(-\frac{22}{3}p - \frac{53}{12} \right)_{j_1} (10p+11)_{j_2} \left(-\frac{16}{9}p - \frac{79}{18} \right)_{j_3} \\ & \quad x_1^{-\frac{22}{3}p - \frac{53}{12} - j_1} \cdot x_2^{10p+11-j_2} \cdot x_3^{-\frac{16}{9}p - \frac{79}{18} - j_3} \end{aligned} \quad (32)$$

For all $x_1 > 0, x_2 \neq 0, x_3 > 0$, $x_1^{11/3} x_2^{-5} x_3^{8/9} > 1$. Therefore, we can determine the 19-th order partial derivative value of $g(x_1, x_2, x_3)$ at $\left(\frac{5}{3}, \frac{1}{4}, \frac{9}{7}\right)$,

$$\begin{aligned} & \frac{\partial^{19} g}{\partial x_3^7 \partial x_2^4 \partial x_1^8} \left(\frac{5}{3}, \frac{1}{4}, \frac{9}{7} \right) \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} \left(-\frac{22}{3}p - \frac{53}{12} \right)_8 (10p+11)_4 \left(-\frac{16}{9}p - \frac{79}{18} \right)_7 \\ & \quad \left(\frac{5}{3} \right)^{-\frac{22}{3}p - \frac{149}{12}} \cdot \left(\frac{1}{4} \right)^{10p+7} \cdot \left(\frac{9}{7} \right)^{-\frac{16}{9}p - \frac{205}{18}} \end{aligned} \quad (33)$$

Also, we use Maple to verify the correctness of (33).

>evalf(D[1\$8,2\$4,3\$7](g)(5/3,1/4,9/7),36);

$$-4.7642925606088240784705 \cdot 10^8$$

>evalf(sum((-1)^p/(2*p+1)*product(-22*p/3-53/12-j,j=0..7)*product(10*p+11-r,r=0..3)*product(-16*p/9-79/18-s,s=0..6)*(5/3)^(-22*p/3-149/12)*(1/4)^(10*p+7)*(9/7)^(-16*p/9-205/18),p=0..infinity),24);

$$-4.76429256060882407847047 \cdot 10^8$$

Case (3). If $x_1^{11/3} x_2^{-5} x_3^{8/9} < -1$. By Case (C) of Theorem 2, we obtain any $j_1 + j_2 + j_3$ -th order partial derivative of $g(x_1, x_2, x_3)$,

$$\begin{aligned} & \frac{\partial^{j_1+j_2+j_3} g}{\partial x_3^{j_3} \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, x_3) \\ &= \pi \cdot \left(-\frac{3}{4} \right)_{j_1} (6)_{j_2} \left(-\frac{7}{2} \right)_{j_3} x_1^{-\frac{3}{4}-j_1} \cdot x_2^{6-j_2} \cdot x_3^{-\frac{7}{2}-j_3} \\ &+ \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} \left(-\frac{22}{3}p - \frac{53}{12} \right)_{j_1} (10p+11)_{j_2} \left(-\frac{16}{9}p - \frac{79}{18} \right)_{j_3} \\ & \quad x_1^{-\frac{22}{3}p - \frac{53}{12} - j_1} \cdot x_2^{10p+11-j_2} \cdot x_3^{-\frac{16}{9}p - \frac{79}{18} - j_3} \end{aligned} \quad (34)$$

For all $x_1 > 0, x_2 \neq 0, x_3 > 0$, $x_1^{11/3} x_2^{-5} x_3^{8/9} < -1$.

Thus, we can evaluate the 21-th order partial derivative value of $g(x_1, x_2, x_3)$ at $\left(\frac{11}{2}, -\frac{5}{8}, \frac{10}{3}\right)$,

$$\begin{aligned} & \frac{\partial^{21} g}{\partial x_3^7 \partial x_2^5 \partial x_1^9} \left(\frac{11}{2}, -\frac{5}{8}, \frac{10}{3} \right) \\ &= \pi \cdot \left(-\frac{3}{4} \right)_9 (6)_5 \left(-\frac{7}{2} \right)_7 \cdot \left(\frac{11}{2} \right)^{-\frac{39}{4}} \cdot \left(-\frac{5}{8} \right) \cdot \left(\frac{10}{3} \right)^{-\frac{21}{2}} \\ &+ \sum_{p=0}^{\infty} \frac{(-1)^p}{2p+1} \left(-\frac{22}{3}p - \frac{53}{12} \right)_9 (10p+11)_5 \left(-\frac{16}{9}p - \frac{79}{18} \right)_7 \times \\ & \quad \left(\frac{11}{2} \right)^{-\frac{22}{3}p - \frac{161}{12}} \cdot \left(-\frac{5}{8} \right)^{10p+6} \cdot \left(\frac{10}{3} \right)^{-\frac{16}{9}p - \frac{205}{18}} \end{aligned} \quad (35)$$

Using Maple to verify the correctness of (35).

>evalf(D[1\$9,2\$5,3\$7](g)(11/2,-5/8,10/3),32);

$$36.0778378678814359$$

>evalf(Pi*product(-3/4-t,t=0..8)*product(6-w,w=0..4)*product(-7/2-s,s=0..6)*(11/2)^(-39/4)*(-5/8)*(10/3)^(-21/2)+sum((-1)^p/(2*p+1)*product(-22*p/3-53/12-j,j=0..8)*product(10*p+11-r,r=0..4)*product(-16*p/9-79/18-u,u=0..6)*(11/2)^(-22*p/3-161/12)*(-5/8)^(10*p+6)*(10/3)^(-16*p/9-205/18),p=0..infinity),32);

$$36.077837867881446$$

4. Conclusion

As mentioned, the evaluation of the partial derivatives of multivariable functions is important. In this study, we propose a new technique to evaluate any order partial derivatives of two types of multivariable functions, and we hope this method can be applied to another multivariable functions. Simultaneously, we know the differentiation term by term theorem plays a significant role in the theoretical inferences of this study. In fact, the application of this theorem is extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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