On Cyclicity and Regularity of Commuting Matrices

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Abstract It is well-known that the following properties of a matrix are equivalent: a matrix is non-derogatory if and only if is cyclic if and only if it is simple and if and only if if it is 1-regular. In this article we attempt to extend these properties to a sequence of commuting matrices and examine the relation between them.

Keywords Commuting matrices, cyclicity, regularity, simplicity, non-derogatory sequences

1 Introduction

Sequences of commuting matrices play an important role in linear algebra (e.g. [9]) as well as its applications to numerical analysis (cf. [1, 19, 18]), algebra (cf. [5]), algebraic geometry (cf. [3, 12, 13]) and approximation theory (cf. [4, 14, 16]). The study of the irreducibility of the variety of commuting couples and triples of matrices was initiated by Motzkin and Taussky [11] and continued in [6], [7] and [8], among others. In this article we will extend some well-known properties of matrices to sequences of commuting matrices and examine their relations to each other.

Our starting point is the following standard fact from linear algebra (cf. [9])

For an $n \times n$ matrix $L$ with complex entries, the following four conditions are equivalent:

R) $L$ is regular, i.e., every eigenspace of $L$ is at most one-dimensional.

C) $L$ is cyclic, i.e., there exists a (cyclic) vector $v \in \mathbb{C}^n$ such that

$$\text{span}\{v, Lv, \ldots, L^{n-1}v\} = \mathbb{C}^n.$$ 

S) $L$ is simple, i.e., if $T$ commutes with $L$ then $T = p(L)$ for some polynomial $p$.

D) $L$ is non-derogatory, i.e., the characteristic polynomial of $L$ is its minimal polynomial.

To what extent these equivalences extend to a sequences of commuting matrices? In this article we will examine the relationship between the four equivalent conditions for $d$-tuple of commuting matrices.

Here are some preliminaries: In what follows, $L(\mathbb{C}^n)$ will stand for the algebra of linear operators on $\mathbb{C}^n$ or, equivalently, for the algebra of complex $n \times n$ matrices. $\mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_d]$ will denote the algebra of polynomials of $d$ variables with complex coefficients. For a subset $F \subset \mathbb{C}^n$ we let $[F]$ stand for the linear span of $F$.

Let $L := (L_1, \ldots, L_d)$ be a sequence of pairwise commuting $n \times n$ matrices with complex entries. A $d$-tuple $\lambda := (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^n$ is called an eigentuple for $L$ if there exists a non-zero vector $v \in \mathbb{C}^n$ such that $L_j v = \lambda_j v$ for all $j = 1, \ldots, d$. Any such vector $v$ is called an eigenvector for $L$ corresponding to an eigentuple $\lambda$. The set of all eigentuples for $L$ is called the (joint) spectrum of $L$ and denoted by $\sigma(L)$. It is well-known and easy to see that $\sigma(L) \neq \emptyset$ for any such $L$.

For $\lambda \in \sigma(L)$ the linear space

$$V_\lambda := \{v \in \mathbb{C}^n : L_j v = \lambda_j v, j = 1, \ldots, d\} \subset \mathbb{C}^n$$

is called an eigenspace for $L$. A subspace $V \subset \mathbb{C}^n$ is $L$-invariant if $L_j V \subset V$ for all $j = 1, \ldots, d$. If $L := (L_1, \ldots, L_d)$ we use $L^*$ to denote the sequence of adjoint matrices $(L_1^*, \ldots, L_d^*)$. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma(L)$ we use $L_\lambda := (L_j - \lambda_j I, j = 1, \ldots, d)$. Finally we use $J_L$ to denote the ideal of polynomials in $\mathbb{C}[x]$ that annihilate $L$:

$$J_L := \{p \in \mathbb{C}[x] : p(L) = 0\}.$$ 

The following useful proposition is a part of the folklore:

Proposition 1.1. Let $U$ be an $L^*$-invariant subspace of $\mathbb{C}^n$. Then $U^\perp$ is an $L$-invariant subspace of $\mathbb{C}^n$.

Proof. For $v \in U^\perp$ and any $u \in U$ we have

$$\langle L_j v, u \rangle = \langle v, L_j^* u \rangle = 0$$

since $L_j^* u \in U$ by our assumption. Hence $L_j v \in U^\perp$. $\square$

Definition 1.2. A $d$-tuple $L := (L_1, \ldots, L_d)$ of pairwise commuting $n \times n$ is called cyclic if there exists a (cyclic) vector $v \in \mathbb{C}^n$ such that $\{p(L)v : p \in \mathbb{C}[x]\} = \mathbb{C}^n$. 

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2 Cyclicity vs. regularity

We start with a simple example (already used in [4], [10]) that shows that the equivalence of cyclicity and regularity fails for pairs of commuting matrices in both directions:

Example 2.1. First consider \( L = (L_1, L_2) \) on \( \mathbb{C}^3 \) given by

\[
L_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\] (2.1)

This is a cyclic commuting pair with the cyclic vector \((1, 0, 0)\), yet \( \sigma(L) = \{(0, 0)\} \) and vectors \((0, 1, 0)\) and \((0, 0, 1)\) are common eigenvectors for \( L \). On the other hand \( L' = (L'_1, L'_2) = (L_1, L_2) = L^* \) given by

\[
L'_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad L'_2 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] (2.2)

is not cyclic (the range of each matrix is the same one-dimensional subspace spanned by \((1, 0, 0)\)), yet the only common eigenspace is one-dimensional, spanned by the vector \( v = (1, 0, 0) \).

It took me a while to learn the lesson of this example: The cyclicity of a \( d \)-tuple of commuting matrices is related to the dimensions of the eigenspaces of the adjoint \( d \)-tuple rather than the matrices themselves. (Of course for one matrix it is a mute point.)

Theorem 2.2. A \( d \)-tuple \( L := (L_1, \ldots, L_d) \) of commuting \( N \times N \) matrices is cyclic iff the dimension of each eigenspace of \( L^* := (L_1^*, \ldots, L_d^*) \) is at most one. In this case the sum of eigenvectors corresponding to distinct eigenvalues of \( L^* \) is a cyclic vector for \( L \).

Theorem 2.2 is an immediate corollary of more general Theorem 2.4 which requires a definition:

Definition 2.3. For a \( d \)-tuple \( L := (L_1, \ldots, L_d) \) of commuting matrices a cyclicity \( \text{cyc}(L) \) is the least integer \( n \) such that there exist \( n \) vectors \( w_1, \ldots, w_n \) with

\[
+_{n=1}^d \{ p(L)w_n : p \in \mathbb{C}[x] \} = \mathbb{C}^n.
\]

If \( \text{cyc}(L) = n \), we will say that \( L \) is \( n \)-cyclic. Thus cyclic \( d \)-tuples are 1-cyclic.

Theorem 2.4. Let \( L := (L_1, \ldots, L_d) \) be a sequence of commuting \( n \times n \) matrices. Then the cyclicity \( \text{cyc}(L) \) is equal to the maximal dimension of eigenspaces of \( L^* \).

Proof. Let \( \{v_1, \ldots, v_s\} \) be the cyclic set for \( s \)-cyclic sequence \( L \) and let \( u_1, \ldots, u_s, u_{s+1} \) be linearly independent eigenvectors that belong to the same eigenspace of \( L^* \). Then there exists a linear combination \( u = \sum_{j=1}^{s+1} \alpha_j u_j \) orthogonal to \( v_1, \ldots, v_s \) (more equations than the unknowns) and \( |h|^2 \) is a proper \( L \)-invariant subspace containing \( v_1, \ldots, v_s \). Contradiction.

Conversely, suppose that \( U_1, \ldots, U_m \) are the eigenspaces of \( L^* \) that correspond to distinct eigenvalues \( \lambda_1, \ldots, \lambda_m \). Let \( s := \max \{ \dim U_j, j = 1, \ldots, m \} \). We will exhibit a set of vectors \( \{w_1, \ldots, w_s\} \) which is a cyclic set for \( L \). For each \( j = 1, \ldots, m \) let \( (u_{1,j}, \ldots, u_{s,j}) \) be vectors in \( U_j \) such that \( (u_{1,j}, \ldots, u_{s,j}) \) are linearly independent if \( k \leq \dim H_j \) and \( u_k,j = 0 \) if \( k > \dim U_j \). Now for \( n = 1, \ldots, s \) we form vectors

\[
w_n := \sum_{j=1}^s u_{n,j}.
\]

We claim that these vectors form a cyclic set for \( L \). Otherwise the space

\[
W := +_{n=1}^s \{ p(L)w_n : p \in \mathbb{C}[x] \}
\]

is a proper \( L \)-invariant subspace of \( \mathbb{C}^n \) hence \( W^\perp \) contains an eigenvector corresponding to some eigenvalue, say \( \lambda_1 \), for \( L \). Let \( p \in \mathbb{C}[x] \) be such that \( p(\lambda_1) = \delta_{1,j} \), for all \( j = 1, \ldots, m \). We have \( p(L^*)w_n = u_{n,1} \); thus \( W \) contains \( U_1 \) and cannot contain an eigenvector from \( U_1 \) orthogonal to it.

Remark 2.5. The second statement in Theorem 2.2 follows directly from the construction of the vectors \( w_n \).

Next, I wish to examine the role that the quadratic polynomials in \( L^* \) play in the cyclicity structure of \( L \).

Since a nilpotent operator is not invertible, its rank is less then the dimension of the space. If \( L_1 \) and \( 0 \neq L_2 \) are two commuting nilpotent operators the range of \( L_2 \) is \( L_1 \)-invariant and \( L_1 \mid _{\text{ran}L_2} \) is still nilpotent; hence the rank \( L_1 L_2 \text{rk} L_2 \) and

\[
\text{dim ker} L_1 L_2 > \text{dim ker} L_2.
\]

In particular for any nilpotent matrix \( L \neq 0 \),

\[
\text{dim ker} L^2 > \text{dim ker} L. \quad (2.3)
\]

Similar result holds for the kernel of sequences of commuting nilpotent matrices; we just need to define the powers of \( L \):

Definition 2.6. Let \( L := (L_1, \ldots, L_d) \) be a sequence of commuting matrices. We define

\[
H_m(L) = \{ p(L) : p \text{ monomials of degree } m \}.
\]

Thus, for instance,

\[
H_2(L_1, L_2) = \{ L_1^2, L_1 L_2, L_2^2 \}.
\]

Also notice that (ordering monomials of degree \( m \)) \( H_m(L) \) is a sequence of commuting matrices.

Lemma 2.7. Let \( 0 \neq L \) be a \( d \)-tuple of commuting nilpotent matrices. Then

\[
\text{dim ker} H_2(L) > \text{dim ker} L.
\]

Proof. Since \( \ker L \subset \ker L_1 \) and by (2.3) above,

\[
\text{dim ker} L^2 > \text{dim ker} L_1 \geq \text{dim ker} L.
\]

Assume that \( k \) is the maximum number of quadratic monomials \( p_1, \ldots, p_k \) such that

\[
\text{dim ker} (p_j(L), j = 1, \ldots, k) > \text{dim ker} L.
\]
Let \( V := \ker (p_j(L), j = 1, \ldots, k) \).

Suppose that a monomial \( L_1 L_m \) is missing from that list. Then \( V \) is invariant for \( L_m \) as well as \( L_i L_m \) and \( \dim \ker (L_i L_m | V) > \dim \ker (L_m | V) \geq \dim \ker L \) since \( \ker L \subset V \). Hence

\[
\dim \ker (L_i L_m, p_j (L), j = 1, \ldots, k) > \dim \ker L.
\]

and thus \( p_1, \ldots, p_k \) are all monomials of degree 2.

The last lemma has an obvious generalization:

**Proposition 2.8.** Let \( L \) be a \( d \)-tuple of commuting nilpotent matrices. If for some \( m \geq 1 \) the set \( H_m (L) \neq \{0\} \) then

\[
\dim \ker H_{m+1} (L) > \dim \ker H_m (L).
\]

Lemma 2.7 has an interesting corollary:

For a matrix \( L \) define

\[
\sqrt{L} := \{ A : A^2 = L \}.
\]

**Corollary 2.9.** For commuting matrices (2.2) from Example 2.1, the sets \( \{L_1, \sqrt{L_2^2} \} \) are not empty yet for any \( A_i \in \sqrt{L_i} \), \( A_1 \) and \( A_2 \) do not commute.

**Proof.** It is easy to verify that

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} \in \sqrt{L_1}, \quad \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \in \sqrt{L_2}
\]

hence \( \sqrt{L_1}, \sqrt{L_2} \) are not empty.

If there exist commuting \( A_i \in \sqrt{L_i} \) then \( \ker (A_1^2, A_2^2) = \ker (L_1, L_2) \) would be at least two dimensional, which is false.

Here are another couple of corollaries of the lemma:

**Corollary 2.10.** If \( 0 \neq L := (L_1, \ldots, L_d) \) is a \( d \)-tuple of commuting matrices then for every \( \lambda := (\lambda_1, \ldots, \lambda_d) \in \sigma (L') \)

\[
\dim \ker ((L_i - \lambda_i I) (L_j - \lambda_j I), i, j = 1, \ldots, d) \geq \dim V_{\lambda}.
\]

**Proof.** The proof follows from Lemma 2.7 and block-diagonalization of commuting matrices [10].

**Corollary 2.11.**

(i) \( L \) is simultaneously diagonalizable iff

\[
\dim \ker (L_{\lambda}, i = 1, \ldots, d) = \dim \ker (H_2 (L_{\lambda}), i = 1, \ldots, d).
\]

(ii) A \( d \)-tuple of commuting matrices has \( N \) distinct eigentuples iff

\[
\dim \ker ((L_i - \lambda_i I)^2, i = 1, \ldots, d) = 1
\]

for every \( \lambda := (\lambda_1, \ldots, \lambda_d) \in \sigma (L') \).

Next we want to address the situation when \( \dim \ker H_2 (L') = 2 \). It follows from the Jordan form that one matrix \( L \) is cyclic if and only if

\[
\dim \ker L^2 = \dim \ker (L')^2 = 2. \quad (2.4)
\]

**Theorem 2.12.** Let \( N \geq 2 \) and \( 0 \neq L \). Then \( \mathcal{A}(L) \) contains a cyclic matrix iff

\[
\dim \ker H_2 (L'_A) = 2
\]

for every \( \lambda \in \sigma (L') \).

**Proof.** It suffices to examine the nilpotent case. If \( L \in \mathcal{A}(L) \) is nilpotent and cyclic then, as follows from the Jordan form of \( L \), \( \dim \ker L^2 = \dim \ker (L')^2 = 2 \). This combined with Lemma 2.7 gives \( \dim \ker H_2 (L') = 2 \).

For the converse, let \( \dim \ker H_2 (L') = 2 \) then \( \dim \ker H_2 (L') = 1 \) and \( L \) is cyclic. I claim that for \( d \) commuting nilpotent matrices

\[
A = (A_1, \ldots, A_d) \quad (2.5)
\]

with \( \ker A = [v_0] \) and \( \dim \ker H_2 (A) = 2 \) at least one of \( A_1 \) is 1-regular. The proof is by induction on \( d \). For \( d = 1 \) the result follows from (2.3). Assume that it is true for \( d - 1 \) and that (2.5) has no cyclic matrices. Then, by inductive assumption, there exist \( u \in \ker (A_2, \ldots, A_d), \) \( u \notin [v_0] \) and \( w \in \ker (A_1, A_2, \ldots, A_{d-1}), w \notin [v_0] \). Let \( k \) and \( m \) be the least integers such that \( A_1^k u = 0 \) and \( A_d^m w = 0 \). I claim that

a) \( 0 \neq A_1^{k-1} u = A_d^{m-1} w \in [v_0] \)

b) Vectors \( v_0, A_1^{k-2} u, A_d^{m-2} w \) are linearly independent.

c) \( v_0, A_1^{k-2} u, A_d^{m-2} w \in \ker H_2 (A) \)

The last two statements contradict \( \dim \ker H_2 (A) = 2 \).

To prove a) we have

\[
A_1^{k-1} u \in \ker A_1 \cap \ker (A_2, \ldots, A_d) = \ker A.
\]

To prove b) assume that for some constants \( \alpha, \beta \) and \( \gamma \)

\[
\alpha v_0 + \beta A_1^{k-2} u + \gamma A_d^{m-2} w = 0.
\]

Then

\[
0 = \alpha A_1 v_0 + \beta A_1^{k-1} u + \gamma A_1 A_d^{m-2} w = \beta v_0
\]

hence \( \beta = 0 \). The argument for \( \gamma \) is the same. Finally c) follows from definitions of \( u, w, k \) and \( m \).

3 Cyclic vs. simple and non-derogatory

We start with a definition of a non-derogatory sequence of commuting matrices. Since the characteristic polynomial of an \( n \times n \) matrix \( L \) is of degree \( n \) hence an equivalent definition of a non-derogatory matrix (cf. D) in the introduction) is \( \dim \mathbb{C}[x]/J_L = n \). Thus the following seem to make sense:

**Definition 3.1.** A \( d \)-tuple \( L := (L_1, \ldots, L_d) \) of commuting \( n \times n \) matrices is non-derogatory if \( \dim (\mathbb{C}[x]/J_L) = n \).

The definition of simplicity is straight forward:

**Definition 3.2.** A \( d \)-tuple \( L := (L_1, \ldots, L_d) \) of commuting \( n \times n \) matrices is simple if every \( T \) that commutes with each \( L_j \) is a polynomial in \( L \).
Proposition 3.3. Let $L := (L_1, \ldots, L_d)$ be a $d$-tuple of commuting $N \times N$ matrices. Then, if $L$ is cyclic it is simple and non-derogatory. Conversely, this is not true.

Proof. Let $T$ commutes with every $L_j$ and let $v$ be a cyclic vector for $L$. Then there exists a polynomial $q \in \mathbb{C}[x]$ such that $q(L)v = T v$. Also for every vector $u \in \mathbb{C}^n$ there exists a polynomial $p_u \in \mathbb{C}[x]$ such that $p_u(L)v = u$. We have

$$Tu = Tp_u(L)v = p_u(L)Tv = p_u(L)q(L)v = q(L)p_u(L)v = q(L)u$$

hence $T = q(L)$.

To prove that a cyclic sequence is non-derogatory we let, once again, $v$ be a cyclic vector for $L$ and define a mapping

$$\varphi : \mathbb{C}[x] \to \mathbb{C}^n$$

$$f \to f(L)v$$

Since $L$ is cyclic, $\varphi$ is onto and by the fundamental theorem of homomorphisms $k[x]/\ker \varphi$ is isomorphic to $\mathbb{C}^n$ hence $\dim \ker \varphi = n$. It remains to show that $\ker \varphi = J_L$. Clearly if $f(L) = 0$ then $f(L)v = 0$. Now assume that $f \in \ker \varphi$, i.e., $f(L)v = 0$. Since for every $u \in \mathbb{C}$ there exists a polynomial $p_u \in \mathbb{C}[x]$ such that $p_u(L)v = u$ we have

$$f(L)u = f(L)p_u(L)v = p_u(L)f(L)v = 0$$

and $f \in J_L$.

To show that the converse fails, we, once more, consider the matrices $L^*$ from Example 2.1. Since $L$ is cyclic $\dim(\mathbb{C}[x,J_L]) = 3$ but $J_L = J_L^*$ hence $L^*$ is non-derogatory yet not cyclic. Similarly, if $T$ commutes with $L_1^*$ and $L_2^*$ then $T^*$ commutes with $L_1$ and $L_2$ and, since $(L_1, L_2)$ is cyclic, it follows that there exists a polynomial $q$ such that $T^* = q(L_1, L_2)$. Hence $T = q(L_1^*, L_2^*)$. Is simple yet non-derogatory.

4 Simple vs. non-derogatory

In this section we will that neither one of the two conditions implies the other.

First we will show that there exists a simple sequence of commuting matrices which is derogatory. In fact we will construct a simple sequence $L$ such that $\dim(\mathbb{C}[x,J_L]) < n$ and a simple sequence $L$ such that $\dim(\mathbb{C}[x,J_L]) > n$.

For the first construction, recall Courter’s example ([2], cf. also [17]) of a commutative subalgebra $A \subset L(\mathbb{C}^{14})$ of dimension 13 which is maximal, i.e., such every matrix that commutes matrices in is in $A$. Let $L := (L_1, \ldots, L_{13})$ be a basis in $A$. Then, by maximality, every matrix that commutes with matrices in $L$ is a (linear homogeneous) polynomial of $L$. Hence $L$ is simple. On the other hand

$$\dim(\mathbb{C}[x_1, \ldots, x_{13}]/J_L) = \dim A = 13 < 14.$$  \hspace{1cm} (4.1)

To see this, consider the 13-dimensional space $H \subset \mathbb{C}[x_1, \ldots, x_{13}]$ of linear homogeneous polynomial. On the other hand for every polynomial $p \in \mathbb{C}[x_1, \ldots, x_{13}]$ the matrix $p(L)$ commutes with $L$, hence there exists $h \in H$ such that $p(L) = h(L)$, hence $p = h + (p - h)$ where $h \in H$ and $(p - h) \in J_L$, i.e.,

$$H + J_L = \mathbb{C}[x_1, \ldots, x_{13}].$$  \hspace{1cm} (4.2)

Since $(L_1, \ldots, L_{13})$ are linearly independent, it follows that $H \cap J_L = \{0\}$, and the sum in (4.2) is a direct sum which proves (4.1).

For the second construction consider a sequence $L$ of four matrices

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (4.3)

It can be verified by direct computations that every matrix that commutes with $L$ a (linear) polynomial in $L$, hence $L$ is simple. On the other hand the identity matrix $I$ is not a linear combination of $L_1, \ldots, L_4$, yet in the algebra

$$A := \{p(L) : p \in \mathbb{C}[x_1, \ldots, x_4]\}.$$ Combined with the fact that $L_j L_k = 0$ for all $j, k = 1, \ldots, 4$ we conclude that $A$ is the space of linear polynomials in four variable, its dimension is $5 > 4$ and, by the same argument as above,

$$\dim(\mathbb{C}[x_1, \ldots, x_4]/J_L) = \dim A = 5 < 4.$$  \hspace{1cm} (4.4)

For the next example consider the sequence $\tilde{L} = (L_1, L_2, L_3)$ consisting of the first three matrices in first (4.3). As was mentioned earlier the pairwise product of these matrices are zero, hence

$$A := \{p(\tilde{L}) : p \in \mathbb{C}[x_1, x_2, x_3]\} = \text{span}\{I, L_1, L_2, L_3\}$$

which is 4-dimensional, hence $\dim(\mathbb{C}[x_1, x_2, x_3]/J_{\tilde{L}}) = 4$ and $\tilde{L}$ is non-derogatory. The fourth matrix in (4.3) commutes with the other three but is not a linear combination of the four matrices in the right-hand side of (4.4). The equality (4.4) thus implies that $L_4$ is not a polynomial in $(I, L_1, L_2, L_3)$ and hence $\tilde{L}$ is not simple.

5 Similarity of commuting sequences

We finish this article with a remark about similarity of commuting $d$-tuples.

Definition 5.1. Two $d$-tuples $L := (L_1, \ldots, L_d)$ and $T := (T_1, \ldots, T_d)$ are similar ($L \sim T$) if there exists an invertible matrix $S$ such that

$$T_j = S L_j S^{-1}$$

for all $j = 1, \ldots, d$.

It is easy to see from the Jordan for that a matrix is it is always similar to its transpose. Example 2.1 shows that it is not the case for a $d$-tuples: $L := (L_1, L_2)$ is not similar to its transpose $L' := (L_1', L_2')$. The following observation was proven in [15]:

Proposition 5.2. A cyclic commuting $d$-tuple $L$ is similar to a commuting $d$-tuple $T$ iff $J_L = J_T$ and $T$ is cyclic. We will now present a direct proof of this fact.
Proof. Let u be a cyclic vector for L and v be a cyclic vector for T. Define a mapping $S : \mathbb{C}^N \to \mathbb{C}^N$ by letting

$$S(p(L)u) = p(T)v$$

for every $p \in \mathbb{C}[x]$. Since u is a cyclic vector for L, $\{p(L)u, p \in \mathbb{C}[x]\} = \mathbb{C}^N$ hence S is defined for all $w \in \mathbb{C}^N$. To show that S is well-defined assume that $p_1(L)u = p_2(L)u$. Then $(p_1(L) - p_2(L))u = 0$ and hence $p_1 - p_2 \in J_L$. By assumption this implies $p_1 - p_2 \in J_T$ thus $p_1(T) - p_2(T) = 0$. In particular $p_1(T)v = p_2(T)v$ and S is well defined and linear. Since v is a cyclic vector for T the map S is onto hence S is invertible. Now let $w \in \mathbb{C}^N$. Then there exists a polynomial p such that $w = p(L)u$. Hence

$$SL_jw = SL_jp(L)u = S(x_jp)(L)u = (x_jp)(T)v = T_jS(p(L)u) = T_jSw.$$ 

Hence $SL_j = T_jS$ for every j and $L \sim T$. \qed

In particular a cyclic commuting d-tuple L is similar to its transpose if and only if $L^T$ is cyclic.

**Problem 5.3.** What is a good criterion for general d-tuple of commuting matrices to be similar to its transpose? Is it sufficient to assume that $\text{cyc}(L) = \text{cyc}(L^T)$?

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