A New Third-Order Derivative Free method for Solving Nonlinear Equations

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Abstract In the present paper, by approximating the derivatives in the Newton-Steffensen third-order method by central difference quotient, we obtain a new modification of this method free from derivatives. We prove that the method obtained preserves their order of convergence, without calculating any derivative. Finally, numerical tests confirm that our method give the better performance as compare to the other well known derivative free Steffensen type methods.

Keywords Nonlinear Equations, Order of Convergence, Simple Root, Central Difference, Derivative Free Method

Mathematics Subject Classification (2000). 65H05, 65H10, 41A25

1 Introduction

A large number of papers have been written about iterative methods for the solution of the nonlinear equations. In this paper, we consider the problem of finding a simple root x^* of a function $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, i.e. $f(x^*) = 0$ and $f'(x^*) \neq 0$. The famous Newton's method for finding x^* uses the iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

starting from some initial value x_0 . The Newton's method is an important and basic method which converges quadratically in some neighborhood of simple root x^* . However, when the first order derivative of the function $f(x)$ is unavailable or is expensive to compute, the Newtons method is still restricted in practical applications. In order to avoid computing the first order derivative, Steffensen in [1] proposed the following derivative-free method. It is well known that the forward-difference approximation for $f'(x)$ at x is

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

If the derivative $f'(x_n)$ is replaced by the forward-difference approximation with $h = f(x_n)$ i.e.

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)},$$

the Newton's method becomes

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)},$$

which is the famous Steffensen's method [1]. The Steffensen's method is based on forward-difference approximation to derivative. This method is a tough competitor of Newton's method. Both the methods are quadratic convergence, both require two functions evaluation per iteration but Steffensen's method is derivative free. The idea of removing derivatives from the iteration process is very significant. Recently, many high order derivative-free methods are built according to the Steffensen's method, see [[2], [3], [4], [5], [6], [7], [8], [9]] and the references therein.

Jain [7] presented the third-order method (Jain Method):

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)^3}{\{f(x_n + f(x_n)) - f(x_n)\}\{f(x_n) - f(y_n)\}}. \end{aligned} \quad (1.1)$$

Dehghan et al. [8] proposed two third-order Steffensen type method (Dehghan Method I):

$$\begin{aligned} y_n &= x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ x_{n+1} &= x_n - \frac{2f(x_n)\{f(x_n) + f(y_n)\}}{f(x_n + f(x_n)) - f(x_n - f(x_n))}. \end{aligned} \quad (1.2)$$

and (Dehghan Method II)

$$\begin{aligned} y_n &= x_n + \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ x_{n+1} &= x_n - \frac{2f(x_n)\{f(y_n) - f(x_n)\}}{f(x_n + f(x_n)) - f(x_n - f(x_n))}. \end{aligned} \quad (1.3)$$

Again Dehghan et al. [9] introduced a new third-order Steffensen type method (Dehghan Method III):

$$\begin{aligned} y_n &= x_n + \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ x_{n+1} &= x_n - \frac{2f(x_n)}{f(y_n)f(u) + f(x_n)f(v)}, \end{aligned} \quad (1.4)$$

where $f(u) = f(x_n + f(x_n)) - f(x_n - f(x_n))$
and $f(v) = f(y_n + f(y_n)) - f(y_n - f(y_n))$.

Recently Cordero et al. [2] presented a fourth-order Steffensen type method (Cordero Method):

$$\begin{aligned} y_n &= x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ x_{n+1} &= x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \cdot \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}. \end{aligned} \quad (1.5)$$

Other Steffensen type methods and their applications are discussed in [[3], [4], [5], [6]].

The purpose of this paper is to develop a new third-order derivative-free method and give the convergence analysis. This paper is organized as follows. In Section 2, we present a new two-step third-order iterative method for solving nonlinear equations. In this method we approximate the derivative of the function by central difference quotient. The new method is free from derivative. We prove that the order of convergence of the new method is three. Numerical examples show better performance of our method in section 4. Section 5 is a short conclusion.

2 Development of the method and analysis of convergence

Let us consider the third-order Newton-Steffensen method [10]:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)^2}{f'(x_n)[f(x_n) - f(y_n)]}. \end{aligned} \quad (2.1)$$

Approximating the derivative $f'(x_n)$ in (2.1) by the central-difference

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n - f(x_n))}{2f(x_n)},$$

we obtain a new method free from derivatives, that we call the *modified Newton-Steffensen method free from derivative (MNSDF)*:

$$\begin{aligned} y_n &= x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ x_{n+1} &= x_n - \frac{2f(x_n)^3}{\{f(x_n + f(x_n)) - f(x_n - f(x_n))\}\{f(x_n) - f(y_n)\}}, \end{aligned} \quad (2.2)$$

Now we are going to prove the method MNSDF has order of convergence three.

Theorem 2.1 Let $x^* \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ in an open interval I . If x_0 is sufficiently close to x^* , then the modified Newton-Steffensen method free from derivative defined by (2.2) has order of convergence three and satisfies the error equation (2.9).

By applying the Taylor series expansion theorem and taking account $f(x^*) = 0$, we can write

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9), \quad (2.3)$$

where $c_k = \frac{f^{(k)}(x^*)}{k!}$, $k = 1, 2, \dots$ and e_n be the error in x_n after n iterations i.e. $e_n = x_n - x^*$;

$$\begin{aligned} f(x_n + f(x_n)) &= (c_1^2 + c_1)e_n + (c_2 c_1^2 + 3c_2 c_1 + c_2)e_n^2 \\ &+ (c_3 c_1^3 + 3c_3 c_1^2 + 2c_1 c_2^2 + 4c_3 c_1 + 2c_2^2 + c_3)e_n^3 \\ &+ (c_4 + c_2(c_2^2 + 2c_1 c_3) + 5c_1 c_4 + 5c_2 c_3 + 6c_1^2 c_4 \\ &\quad + 4c_1^3 c_4 + c_1^4 c_4 + 6c_1 c_2 c_3 + 3c_1^2 c_2 c_3)e_n^4 \\ &+ O(e_n^5). \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} f(x_n - f(x_n)) &= (-c_1^2 + c_1)e_n + (c_2 c_1^2 - 3c_2 c_1 + c_2)e_n^2 \\ &+ (-c_3 c_1^3 + 3c_3 c_1^2 + 2c_1 c_2^2 - 4c_3 c_1 - 2c_2^2 + c_3)e_n^3 \\ &+ (c_4 + c_2(c_2^2 + 2c_1 c_3) - 5c_1 c_4 - 5c_2 c_3 + 6c_1^2 c_4 \\ &\quad - 4c_1^3 c_4 + c_1^4 c_4 + 6c_1 c_2 c_3 - 3c_1^2 c_2 c_3)e_n^4 \\ &+ O(e_n^5). \end{aligned} \quad (2.5)$$

Further more it can be easily find

$$\begin{aligned} &\frac{2.f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \\ &= e_n + (-c_2/c_1)e_n^2 + ((2c_2^2)/c_1^2 - (2c_3)/c_1 - c_1 c_3)e_n^3 \\ &\quad + (c_2 c_3 - 4c_1 c_4 - (3c_4)/c_1 - (4c_2^3)/c_1^3 + (7c_2 c_3)/c_1^2)e_n^4 \\ &\quad + O(e_n^5). \end{aligned} \quad (2.6)$$

By considering this relation and expression of y_n in the equation (2.2), we obtain

$$\begin{aligned} y_n &= x^* + (c_2/c_1)e_n^2 - ((2c_2^2)/c_1^2 - (2c_3)/c_1 - c_1 c_3)e_n^3 \\ &\quad - (c_2 c_3 - 4c_1 c_4 - (3c_4)/c_1 - (4c_2^3)/c_1^3 + (7c_2 c_3)/c_1^2)e_n^4 \\ &\quad + O(e_n^5). \end{aligned} \quad (2.7)$$

At this time, we should expand $f(y_n)$ around the root by taking into consideration (2.7). Accordingly, we have

$$\begin{aligned} f(y_n) &= c_2 e_n^2 + c_1(c_1 c_3 + (2c_3)/c_1 - (2c_2^2)/c_1^2)e_n^3 \\ &\quad + (c_1(4c_1 c_4 - c_2 c_3 + (3c_4)/c_1 + (4c_2^3)/c_1^3 - (7c_2 c_3)/c_1^2) + c_2^3/c_1^2)e_n^4 \\ &\quad + O(e_n^5). \end{aligned} \quad (2.8)$$

By using (2.3), (2.8), (2.4) and (2.5) in the last expression of (2.2), we obtain

$$e_{n+1} = (c_2^2/c_1^2)e_n^3 + O(e_n^4). \quad (2.9)$$

3 Numerical Tests

In this section, in order to compare the our new method with Steffensen method, Jain method, Dehghan method I, Dehghan method II, Dehghan method III and Cordero method, we give some numerical examples. For this consider the following functions:

Table 1. Test functions and their roots.

Non-linear functions	Roots
$f_1(x) = \sin^2(x) - x^2 + 1$	1.404492
$f_2(x) = x^2 - e^x - 3x + 2$	0.257530
$f_3(x) = \cos(x) - x$	0.739085
$f_4(x) = \cos(x) - xe^x + x^2$	0.639154
$f_5(x) = e^x - 1.5 - \arctan(x)$	0.767653

Table 2-6 shows the comparison of these methods for these functions. All the numerical computations have been carried out using MATHEMATICA 8. The numerical results show that the our proposed method is efficient.

Table 2. Errors Occurring in the estimates of the root of function $f_1(x) = \sin^2(x) - x^2 + 1$ after third iteration by the method described with initial guess $x_0 = 1$.

Methods	$ f_1(x_3) $
<i>Steffensen Method</i>	0.27307e-3
<i>Jain Method</i>	0.15974e-11
<i>Dehghan I Method</i>	0.12208e-3
<i>Dehghan II Method</i>	0.27058e-7
<i>Dehghan III Method</i>	0.23473e-9
<i>Cordero Method</i>	0.16292e-9
<i>MNSDF</i>	0.22293e-12

Table 3. Errors Occurring in the estimates of the root of function $f_2(x) = x^2 - e^x - 3x + 2$ after third iteration by the method described with initial guess $x_0 = 0.7$.

Methods	$ f_2(x_3) $
<i>Steffensen Method</i>	0.79151e-6
<i>Jain Method</i>	0.99771e-31
<i>Dehghan I Method</i>	0.27162e-5
<i>Dehghan II Method</i>	0.20573e-22
<i>Dehghan III Method</i>	0.70112e-12
<i>Cordero</i>	0.89687e-6
<i>MNSDF</i>	0.82122e-31

Table 4. Errors Occurring in the estimates of the root of function $f_3(x) = \cos(x) - x$ after third iteration by the method described with initial guess $x_0 = 1$.

Methods	$ f_3(x_3) $
<i>Steffensen Method</i>	0.82149e-10
<i>Jain Method</i>	0.10235e-34
<i>Dehghan I Method</i>	0.17099e-23
<i>Dehghan II Method</i>	0.61489e-33
<i>Dehghan III Method</i>	0.19366e-28
<i>Cordero Method</i>	0.14803e-15
<i>MNSDF</i>	0.98665e-36

Table 5. Errors Occurring in the estimates of the root of function $f_4(x) = \cos(x) - xe^x + x^2$ after third iteration by the method described with initial guess $x_0 = 1$.

Methods	$ f_4(x_3) $
<i>Steffensen Method</i>	0.46704e-2
<i>Jain Method</i>	0.15915e-10
<i>Dehghan I Method</i>	0.85626e-2
<i>Dehghan II Method</i>	0.40435e-7
<i>Dehghan III Method</i>	0.15970e-5
<i>Cordero Method</i>	0.19461e-2
<i>MNSDF</i>	0.53152e-12

Table 6. Errors Occurring in the estimates of the root of function $f_5(x) = e^x - 1.5 - \arctan(x)$ after third iteration by the method described with initial guess $x_0 = 1$.

Methods	$ f_5(x_3) $
<i>Steffensen Method</i>	0.78069e-3
<i>Jain Method</i>	0.99920e-15
<i>Dehghan I Method</i>	0.43288e+0
<i>Dehghan II Method</i>	0.11102e-15
<i>Dehghan III Method</i>	0.21494e-10
<i>CorderoMethod</i>	0.55888e-13
<i>MNSDF</i>	0.11102e-15

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