A Chiellini Type Integrability Condition for the Generalized First Kind Abel Differential Equation

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Abstract The Chiellini integrability condition of the first order first kind Abel equation $dy/dx = f(x)y^2 + g(x)y^3$ is extended to the case of the general Abel equation of the form $dy/dx = a(x) + b(x)y + f(x)y^{\alpha-1} + g(x)y^\alpha$, where $\alpha \in \mathbb{R}$, and $\alpha > 1$. In the case $\alpha = 2$ the generalized Abel equations reduces to a Riccati type equation, for which a Chiellini type integrability condition is obtained.

Keywords First Kind Abel Differential Equation, Generalized Abel Differential Equation, Riccati Equation, Integrability Conditions, Exact Solutions

An integrability condition for the Abel Eq. (1) was obtained by Chiellini [3, 4], which can be formulated as follows [3]:

Chiellini integrability condition. A first kind Abel type differential equation of the form given by Eq. (1) can be exactly integrated if the functions $f(x)$ and $g(x)$ satisfy the condition

$$\frac{d}{dx} \frac{g(x)}{f(x)} = kf(x), \quad k = \text{constant}, \quad k \neq 0. \quad (5)$$

This integrability result for the Abel equation has been applied for obtaining exact solutions of second order differential equations that can be reduced to an Abel type equation in [5] - [12].

The Abel Eq. (1) is the “reduced” form of the general first kind Abel equation $dy/dx = a(x) + b(x)y + f(x)y^2 + g(x)y^3$, where $a(x), b(x), f(x), g(x)$ are arbitrary real functions. A possibility of generalizing the reduced Abel Eq. (1) is to consider that the non-linear terms in y (the term y^2 and y^3 , respectively), appearing in the right hand side of the general Abel equation are at some arbitrary powers $y^{\alpha-1}$ and y^α , respectively, where $\alpha > 1$ is a real number. We call this equation a generalized first kind Abel equation.

It is the purpose of the present paper to extend the Chiellini integrability condition to the generalized Abel equations of the first kind. If the coefficients of the equation satisfy two integrability conditions, the general solution can be found through quadratures. In the particular case $\alpha = 2$, the generalized Abel equations reduces to a Riccati type equation, and a Chiellini type integrability condition for the Riccati equation is obtained.

The present paper is organized as follows. The integrability condition for the generalized Abel equation is obtained in Section 2. The Chiellini type integrability condition for the Riccati equation is obtained in Section 3. We conclude our results in Section 4.

1 Introduction

The third degree polynomial, first kind first order Abel differential equation [1]

$$\frac{dy}{dx} = f(x)y^2 + g(x)y^3, \quad (1)$$

where the coefficients $f(x)$ and $g(x)$ are real functions of the variable x , plays an important role in many physical and mathematical problems. For example, the Liénard type second order differential equation, given by [2]

$$\frac{d^2y}{dx^2} + f(y) \frac{dy}{dx} + g(y) = 0, \quad (2)$$

can be transformed to the Liénard system, represented by the autonomous system of differential equations

$$\frac{dy}{dx} = u, \quad \frac{du}{dx} = -f(y)u - g(y), \quad (3)$$

describing nonlinear damped oscillations of a dynamical system. With the help of the substitution $u = 1/v$, the Liénard system can easily be reduced to a first-order first kind Abel differential equation of the form

$$\frac{dv}{dy} = f(y)v^2 + g(y)v^3. \quad (4)$$

2 The Chiellini integrability condition for generalized Abel equations

We introduce a generalization of the Abel first kind differential Eq. (1) through the following:

Definition. The first order nonlinear ordinary differential equation, which can be given as

$$\frac{dy}{dx} = a(x) + b(x)y + f(x)y^{\alpha-1} + g(x)y^\alpha, \quad \alpha \in \mathbb{R}, \quad \alpha > 1, \quad (6)$$

where $a(x), b(x), f(x), g(x) \in C^\infty(I)$ are arbitrary functions defined on a real interval $I \subseteq \mathbb{R}$, $a(x), b(x), f(x), g(x) \neq 0, \forall x \in I$, and $\alpha \in \mathbb{R}$ satisfies the condition $\alpha > 1$, is called the generalized first kind Abel type differential equation.

By introducing a new function $p(x)$, defined as

$$y(x) = e^{\int b(x)dx} p(x), \quad (7)$$

Eq. (6) becomes

$$\begin{aligned} \frac{dp(x)}{dx} &= a(x)e^{-\int b(x)dx} + f(x)e^{(\alpha-2)\int b(x)dx} p^{\alpha-1}(x) \\ &\quad + g(x)e^{(\alpha-1)\int b(x)dx} p^\alpha(x). \end{aligned} \quad (8)$$

We assume now that the functions $b(x)$, $f(x)$ and $g(x)$ satisfy the generalized Chiellini condition

$$\begin{aligned} \frac{d}{dx} \left[\frac{f(x)}{g(x)e^{\int b(x)dx}} \right] &= -k_1 \frac{f^\alpha(x)}{g^{\alpha-1}(x)} e^{-\int b(x)dx}, \\ k_1 &= \text{constant}, \quad k_1 \neq 0, \end{aligned} \quad (9)$$

where $k_1 \in \mathbb{R}$, $k_1 \neq 0$, is an arbitrary constant. Then, by introducing the transformation

$$p(x) = \frac{f(x)}{g(x)e^{\int b(x)dx}} s(x), \quad (10)$$

Eq. (8) becomes

$$\frac{ds}{dx} = \frac{g(x)}{f(x)} a(x) + \frac{f^{\alpha-1}(x)}{g^{\alpha-2}(x)} (s^\alpha + s^{\alpha-1} + k_1 s). \quad (11)$$

Therefore we have obtained the following generalization of the Chiellini integrability condition:

Theorem 1. If the coefficients $a(x)$, $b(x)$, $f(x)$ and $g(x)$ of the generalized first kind Abel Eq. (6) satisfy the conditions (9) and

$$a(x) = k_2 \frac{f^\alpha(x)}{g^{\alpha-1}(x)} = -\frac{k_2}{k_1} e^{\int b(x)dx} \frac{d}{dx} \left[\frac{f(x)}{g(x)e^{\int b(x)dx}} \right], \quad (12)$$

respectively, where $k_2 \in \mathbb{R}$, $k_2 \neq 0$, is an arbitrary constant, the general solution of Eq. (6) is given by

$$y(x) = \frac{f(x)}{g(x)} s(x), \quad (13)$$

where $s(x)$ is a solution of the equation

$$\left| \frac{g(x)e^{\int b(x)dx}}{f(x)} \right| = K^{-1} e^{F[s(x), k_1, k_2]}, \quad (14)$$

$K \neq 0$ is an arbitrary integration constant, and

$$F[s(x), k_1, k_2] = k_1 \int \frac{ds}{s^\alpha + s^{\alpha-1} + k_1 s + k_2}. \quad (15)$$

In order to obtain the above Theorem we have used the result, which represents the generalization of the transformations introduced for the third degree Abel polynomial equation in [10],

$$\begin{aligned} \int \frac{f^{\alpha-1}(x)}{g^{\alpha-2}(x)} dx &= \\ -\frac{1}{k_1} \int \frac{1}{f(x)/g(x)e^{\int b(x)dx}} d \left[\frac{f(x)}{g(x)e^{\int b(x)dx}} \right] &= \\ \frac{1}{k_1} \ln \left| \frac{g(x)e^{\int b(x)dx}}{f(x)} \right| + K_0, \end{aligned} \quad (16)$$

where $K_0 \neq 0$ is an arbitrary constant of integration, which is related to the constant K by the relation $K = \exp(k_1 K_0)$.

The generalized Chiellini integrability condition, given by Eq. (9), can be reformulated as a Bernoulli type differential equation for $f(x)$,

$$\frac{df}{dx} = \left[\frac{1}{g(x)} \frac{dg(x)}{dx} + b(x) \right] f(x) - k_1 \frac{f^\alpha(x)}{g^{\alpha-2}(x)}, \quad (17)$$

with the general solution given by

$$\begin{aligned} f(x) &= g(x)e^{\int b(x)dx} \left[K_1 - (1-\alpha)k_1 \times \right. \\ &\quad \left. \int e^{(\alpha-1)\int b(x)dx} g(x)dx \right]^{1/(1-\alpha)}, \quad \alpha \neq 1, \end{aligned} \quad (18)$$

where K_1 is an arbitrary constant of integration.

As a differential equation for $g(x)$, the integrability condition Eq. (9) can be reformulated as

$$g^{\alpha-3}(x) \frac{dg}{dx} = \left[\frac{1}{f(x)} \frac{df(x)}{dx} - b(x) \right] g^{\alpha-2}(x) + k_1 f^{\alpha-1}(x), \quad (19)$$

with the general solution given by

$$\begin{aligned} g(x) &= f(x)e^{-\int b(x)dx} \left[K_2 + (\alpha-2)k_1 \times \right. \\ &\quad \left. \int f(x)e^{(\alpha-2)\int b(x)dx} dx \right]^{1/(\alpha-2)}, \quad \alpha \neq 2, \end{aligned} \quad (20)$$

where K_2 is an arbitrary constant of integration.

Therefore we have obtained the following:

Theorem 2. If the coefficients $a(x)$, $b(x)$, $f(x)$ and $g(x)$ of the generalized second type Abel differential Eq. (6) satisfy the conditions (18), or (20), and (12), then the differential equation is exactly integrable, and its general solution can be obtained from Eqs. (13)-(15).

Theorem 2 represents the generalization to the case of the generalized Abel equation of the results obtained initially for the third degree Abel differential equation in [10], and further discussed in [12].

A particular case of the integrability of the generalized Abel equation can be obtained by assuming that the constants k_1 and k_2 satisfy the condition $k_1 = k_2$. Then the function $F[s(x), k_1, k_2]$ can be written as

$$F[s(x), k_1, k_2] = k_1 \int \frac{ds}{(s^{\alpha-1} + k_1)(s+1)}. \quad (21)$$

In many cases the above integral can be obtained exactly. As a particular case we consider the value $\alpha = 3/2$. Hence the generalized Abel type equation takes the form

$$\frac{dy}{dx} = a(x) + b(x)y + f(x)y^{1/2} + g(x)y^{3/2}. \quad (22)$$

Therefore if the coefficients $a(x)$, $b(x)$, $f(x)$ and $g(x)$ of the generalized Abel differential Eq. (22) satisfy the conditions

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)e^{\int b(x)dx}} \right] = -k_1 \frac{f^{3/2}(x)}{g^{1/2}(x)} e^{-\int b(x)dx}, \quad (23)$$

and

$$a(x) = k_1 \frac{f^{3/2}(x)}{g^{1/2}(x)}, \quad (24)$$

respectively, then the solution of Eq. (22) is given by

$$y(x) = \frac{f(x)}{g(x)} s(x), \quad (25)$$

where $s(x)$ is a solution of the equation

$$K \left| \frac{g(x)e^{\int b(x)dx}}{f(x)} \right| = e^{k_1 \int \frac{ds}{(\sqrt{s}+k_1)(s+1)}} = \frac{(1+s)^{k_1^2/(1+k_1^2)}}{(k_1 + \sqrt{s})^{2k_1^2/(1+k_1^2)}} e^{[2k_1/(1+k_1^2)] \arctan \sqrt{s}}. \quad (26)$$

By power expanding the right hand side of Eq. (26) one can easily obtain some approximate solutions of Eq. (22).

3 The generalized Chiellini integrability condition for the Riccati equation

In the case $\alpha = 2$ the generalized second type Abel differential Eq. (6) reduces to a Riccati type equation [1] of the form

$$\frac{dy}{dx} = a(x) + [b(x) + f(x)]y + g(x)y^2. \quad (27)$$

Therefore the generalized Chiellini condition can be extended to the case of Eq. (27), giving the following integrability condition for the Riccati equation:

Theorem 3. If the coefficients $a(x)$, $f(x)$, $b(x)$ and $g(x)$ of the Riccati Eq. (27) satisfy the conditions

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)e^{\int b(x)dx}} \right] = -k_1 \frac{f^2(x)}{g(x)} e^{-\int b(x)dx}, \quad k_1 = \text{constant}, \quad k_1 \neq 0, \quad (28)$$

and

$$a(x) = k_2 \frac{f^2(x)}{g(x)}, \quad k_2 = \text{constant}, \quad k_2 \neq 0, \quad (29)$$

respectively, where $k_1, k_2 \in \mathbb{R}$, $k_1 \neq 0$, $k_2 \neq 0$, are arbitrary constants, then the general solution of the Riccati Eq. (27) is given by

$$y(x) = \frac{f(x)}{g(x)} \left\{ \frac{\sqrt{-\Delta}}{2} \tan \left[\frac{\sqrt{-\Delta}}{2k_1} \ln \left| \frac{Kg(x)e^{\int b(x)dx}}{f(x)} \right| \right] - \frac{1+k_1}{2} \right\}, \quad \Delta < 0, \quad (30)$$

$$y(x) = -\frac{f(x)}{g(x)} \left\{ \frac{k_1}{\ln |Kg(x)e^{\int b(x)dx}/f(x)|} + \frac{1+k_1}{2} \right\}, \quad \Delta = 0, \quad (31)$$

$$y(x) = -\frac{f(x)}{g(x)} \left\{ \frac{\sqrt{\Delta}}{2} \tanh \left[\frac{\sqrt{\Delta}}{2k_1} \ln \left| \frac{Kg(x)e^{\int b(x)dx}}{f(x)} \right| \right] + \frac{1+k_1}{2} \right\}, \quad \Delta > 0, \quad (32)$$

where K is an arbitrary integration constant, and $\Delta = (1+k_1)^2 - 4k_2$.

The integrability condition of the Riccati Eq. (28) can be written as a differential equation for $f(x)$,

$$\frac{df(x)}{dx} = \left[\frac{1}{g(x)} \frac{dg(x)}{dx} + b(x) \right] f(x) - k_1 f^2(x), \quad (33)$$

with the general solution given by

$$f(x) = \frac{g(x)e^{\int b(x)dx}}{K_3 + k_1 \int g(x) \exp \left[\int b(x) dx \right] dx}, \quad (34)$$

where K_3 is an arbitrary integration constant. As a differential equation for $g(x)$, the Riccati equation integrability condition (28) can be formulated as

$$\frac{dg(x)}{dx} = \left[\frac{1}{f(x)} \frac{df(x)}{dx} - b(x) + k_1 f(x) \right] g(x), \quad (35)$$

$$g(x) = K_4 f(x) e^{-\int [b(x) - k_1 f(x)] dx}, \quad (36)$$

where K_4 is an arbitrary integration constant.

Therefore we have obtained the following:

Theorem 4. If the coefficients $a(x)$, $f(x)$, $b(x)$ and $g(x)$ of the Riccati equation (27) satisfy the conditions (34), or (36), and (29), then the Riccati equation is exactly integrable, and its general solution is given by Eqs. (30)-(32).

4 Concluding remarks

In the present paper, we have obtained an exact integrability condition for the generalized first kind Abel equation. The general solution of the generalized nonlinear Abel differential equation can be obtained by quadratures if the four coefficients of the equation satisfy two consistency conditions. The constraint imposes

severe restrictions, limiting the number of possible solutions that can be obtained in this way. An integrability condition for the Riccati equation, representing a particular case of the generalized Abel equation, was also obtained. Some applications of the present formalism to the Abel equations appearing in physical problems will be presented in a future paper.

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