Triple Coincidence Point Theorems for Multi-Valued Maps in Partially Ordered Metric Spaces

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Abstract In this paper we prove a triple coincidence point theorem for multi-valued and single-valued mappings in a partially ordered metric space based on the concepts of [5]. Also we give an example which supports our main result. Our result generalizes several results relating to coupled fixed point theorems.

Keywords Triple fixed point, complete space, w-compatible, set-valued mapping, ∆-Symmetric Property

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1 Introduction

The study of fixed points for multi-valued contraction mappings using the Hausdorff metric was initiated by Nadler [9].

Let \((X, d)\) be a metric space. We denote \(CB(X)\) the family of all non-empty closed and bounded sub sets of \(X\) and \(CL(X)\) the set of all non-empty closed sub sets of \(X\). For \(A, B \in CB(X)\) and \(x \in X\), we denote \(D(x, A) = \inf \{d(x, a) : a \in A\}\). Let \(H\) be the Hausdorff metric induced by the metric \(d\) on \(X\), that is

\[H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}\]

for every \(A, B \in CB(X)\).

Definition 1.1 An element \(x \in X\) is said to be a fixed point of a set-valued mapping \(T : X \rightarrow CB(X)\) if and only if \(x \in Tx\).

In 1969, Nadler [9] extended the famous Banach Contraction Principle [8] from single-valued mapping to multi-valued mapping and proved the following fixed point theorem for the multi-valued contraction.

Theorem 1.2 ([9]): Let \((X, d)\) be a complete metric space and let \(T\) be a mapping from \(X\) into \(CB(X)\). Assume that there exists \(c \in [0, 1)\) such that

\[H(Tx, Ty) \leq c \, d(x, y),\]

for all \(x, y \in X\). Then, \(T\) has a fixed point.

The existence of fixed points for various multi-valued contractive mappings has been studied by many authors under different conditions. For details, we refer the reader to [1, 4, 7, 9, 11] and the references therein.

The concept of coupled fixed point for multi-valued mapping was introduced by Samet and Vetro [2] and later several authors namely Hussain and Alotaibi [6] and Aydi et. al. [3] proved coupled coincidence point theorems in partially ordered metric spaces.

Berinde and Borcut [10], introduced the concept of triple fixed points and obtained a tripped fixed point theorem for single valued map. Later we introduced Triple fixed, Triple coincidence and Triple common fixed points for multi-valued maps in our earlier paper [5] as follows.
Definition 1.3 ([5]) Let $X$ be a non empty set, $T : X \times X \times X \to 2^X$ (Collection of all non empty subsets of $X$). $f : X \to X$.

(i) The point $(x, y, z) \in X \times X \times X$ is called a tripled fixed of $T$ if

\[ x \in T(x, y, z), \quad y \in T(y, x, y) \quad \text{and} \quad z \in T(z, y, x). \]

(ii) The point $(x, y, z) \in X \times X \times X$ is called a tripled coincident point of $T$ and $f$ if

\[ fx \in T(x, y, z), \quad fy \in T(y, x, y) \quad \text{and} \quad fz \in T(z, y, x). \]

(iii) The point $(x, y, z) \in X \times X \times X$ is called a tripled common fixed point of $T$ and $f$ if

\[ x = fx \in T(x, y, z), \quad y = fy \in T(y, x, y) \quad \text{and} \quad z = fz \in T(z, y, x). \]

Definition 1.4 ([5]) Let $T : X \times X \times X \to 2^X$ be a multi - valued map and $f$ be a self map on $X$. The Hybrid pair \{T, f\} is called $w$ - compatible if $T(f(x, y, z)) \subseteq T(fx, fy, fz)$ whenever $(x, y, z)$ is a tripled coincidence point of $T$ and $f$.

2 Results

Let $(X, d)$ be a metric space endowed with a partial order $\preceq$ and $G : X \to X$. Define the set $\Delta \subset X^3$ by

\[ \Delta = \{(x, y, z) \in X^3 : \text{Gx} \cap \text{Gy} \neq \emptyset \}. \]

Definition 2.1 A mapping $F : X^3 \to X$ is said to have a $\Delta$ - Symmetric property if and only if $(x, y, z) \in \Delta \Rightarrow F(x, y, z)RF(y, x, y), F(y, x, y)RF(z, y, x)$ and $F(y, x, y)RF(z, y, x)$.

Definition 2.2 A mapping $f : X^3 \to [0, \infty)$ is called lower semi continuous if, for any sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in $X$ and $(x, y, z) \in X^3$, one has

\[ \lim_{n \to \infty} (x_n, y_n, z_n) = (x, y, z) \implies f(x, y, z) \leq \liminf_{n \to \infty} f(x_n, y_n, z_n). \]

Theorem 2.3 Let $(X, d)$ be a metric space endowed with a partial order $\preceq$ and $\Delta \neq \emptyset$. Suppose that $F : X \times X \times X \to CL(X)$ has a $\Delta$ - Symmetric property, $g : X \to X$ is continuous, $g(X)$ is complete, the function $f : g(X) \times g(X) \times g(X) \to [0, +\infty)$ defined for all $x, y, z \in X$ by

\[ (2.3.1) \quad f(gx, gy, gz) := D(gx, F(x, y, z)) + D(gy, F(y, x, y)) + D(gz, F(z, y, x)), \]

is lower semi - continuous and there exists a function $\phi : [0, +\infty) \to [a, 1]$, $0 < a < 1$, satisfying

\[ (2.3.2) \quad \lim_{r \to +\infty} \sup_{x \in [0, +\infty)} \phi(r) < 1, \text{ for each } x \in [0, +\infty). \]

Assume that for any $(x, y, z) \in \Delta$ there exist $gu \in F(x, y, z), gv \in F(y, x, y)$ and $gw \in F(z, y, x)$ satisfying

\[ (2.3.3) \quad \sqrt{\phi(f(gx, gy, gz))}[d(gx, gu) + d(gy, gv) + d(gz, gw)] \leq f(gx, gy, gz) \]

such that

\[ (2.3.4) \quad f(gu, gv, gw) \leq \phi(f(gx, gy, gz))[d(gx, gu) + d(gy, gv) + d(gz, gw)]. \]

Then, $F$ and $g$ have a triple coincidence point. That is, there exists $(go, gb, g\gamma) \in X \times X \times X$ such that $go \in F(\alpha, \beta, \gamma), gb \in F(\beta, \alpha, \gamma)$ and $g\gamma \in F(\gamma, \beta, \alpha)$.

Let $(x_0, y_0, z_0) \in \Delta$ be arbitrary and fixed, by (2.3.3) and (2.3.4), we can choose $gx_1 \in F(x_0, y_0, z_0), gy_1 \in F(y_0, x_0, y_0)$ and $gz_1 \in F(z_0, y_0, x_0)$ such that

\[ \sqrt{\phi(f(gx_0, gy_0, gz_0))}[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)] \leq f(gx_0, gy_0, gz_0) \quad (2.1) \]

and

\[ f(gx_1, gy_1, gz_1) \leq \phi(f(gx_0, gy_0, gz_0))[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)]. \quad (2.2) \]

From (2.1) and (2.2), we get

\[ f(gx_1, gy_1, gz_1) \leq \phi(f(gx_0, gy_0, gz_0))[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)] \]

\[ = \sqrt{\phi(f(gx_0, gy_0, gz_0))} \left[ \sqrt{\phi(f(gx_0, gy_0, gz_0))} \right] \]

\[ \leq \sqrt{\phi(f(gx_0, gy_0, gz_0))} f(gx_0, gy_0, gz_0). \]
Thus
\[ f(gx_1, gy_1, gz_1) \leq \sqrt{\phi(fgx_0, gy_0, gz_0)}f(gx_0, gy_0, gz_0). \] (2.3)

Now, since F has a $\Delta$-Symmetric property and $(x_0, y_0, z_0) \in \Delta$, we have
\[ F(x_0, y_0, z_0)RF(y_0, x_0, y_0), F(x_0, y_0, z_0)RF(z_0, y_0, x_0) \]
and
\[ F(y_0, x_0, y_0)RF(z_0, y_0, x_0) \]
Thus
\[ gx_1 R gy_1, gx x_1 R gz_1 \text{ and } gy_1 R gz_1 \] (2.4)

By definition of $\Delta$, $(x_1, y_1, z_1) \in \Delta$.

Again from (2.3.3) and (2.3.4), we can choose $gx_2 \in F(x_1, y_1, z_1)$, $gy_2 \in F(y_1, x_1, y_1)$ and $gz_2 \in F(z_1, y_1, x_1)$ such that
\[ \sqrt{\phi(fgx_1, gy_1, gz_1)} [d(gx_1, gx_2) + d(gy_1, gy_2) + d(gz_1, gz_2)] \leq f(gx_1, gy_1, gz_1) \]
and
\[ f(gx_2, gy_2, gz_2) \leq \phi(f(gx_1, gy_1, gz_1)) [d(gx_1, gx_2) + d(gy_1, gy_2) + d(gz_1, gz_2)] . \]

Hence, we get
\[ f(gx_2, gy_2, gz_2) \leq \sqrt{\phi(fgx_1, gy_1, gz_1)}f(gx_1, gy_1, gz_1), \]
with $(x_2, y_2, z_2) \in \Delta$.

Continuing this process we can choose $gx_n \in X, gy_n \in X$ and $gz_n \in X$ such that for all $n = 0, 1, 2, \ldots$, we have $(x_n, y_n, z_n) \in \Delta$, $gx_{n+1} \in F(x_n, y_n, z_n)$, $gy_{n+1} \in F(y_n, x_n, y_n)$ and $gz_{n+1} \in F(z_n, y_n, x_n)$,
\[ \sqrt{\phi(fgx_n, gy_n, gz_n)} \leq \sup \phi(fgx_n, gy_n, gz_n) \] (2.5)

and
\[ f(gx_{n+1}, gy_{n+1}, gz_{n+1}) \leq \sqrt{\phi(fgx_n, gy_n, gz_n)}f(gx_n, gy_n, gz_n), \] (2.6)

with $(x_{n+1}, y_{n+1}, z_{n+1}) \in \Delta$.

Now, we shall show that $f(gx_n, gy_n, gz_n) \to 0$ as $n \to \infty$.

If $f(gx_m, gy_m, gz_m) = 0$, for some $m$, then we get
\[ D(gx_m, F(x_m, y_m, z_m)) = 0, \text{ implies that } gx_m \in F(x_m, y_m, z_m), \]
\[ D(gy_m, F(y_m, x_m, y_m)) = 0, \text{ implies that } gy_m \in F(y_m, x_m, y_m) \]
and
\[ D(gz_m, F(z_m, y_m, x_m)) = 0, \text{ implies that } gz_m \in F(z_m, y_m, x_m). \]

Hence in this case $(gx_m, gy_m, gz_m)$ is a triple coincidence point of $F$ and $g$ and the theorem is proved.

Suppose that $f(gx_n, gy_n, gz_n) > 0$ for all $n$.

Using (2.7) and $\phi(t) < 1$, we conclude that \{\{gx_n, gy_n, gz_n\}\} is a strictly decreasing sequence of non-negative real numbers. Thus, there exists a $\delta > 0$, such that
\[ \lim_{n \to \infty} f(gx_n, gy_n, gz_n) = \delta. \]

Now, we will show that $\delta = 0$. On contrary assume that $\delta > 0$.

Letting $n \to \infty$ in (2.7), we have that
\[ \delta \leq \lim_{f(gx_n, gy_n, gz_n) \to \delta^+} \sup \sqrt{\phi(fgx_n, gy_n, gz_n)} \delta < \delta, \]
a contradiction. Hence $\delta = 0$. That is
\[ \lim_{n \to \infty} f(gx_n, gy_n, gz_n) = 0. \] (2.8)

Now, we prove \{gx_n\}, \{gy_n\} and \{gz_n\} are Cauchy sequences in $(X, d)$.

Suppose
\[ \delta \leq \lim_{f(gx_n, gy_n, gz_n) \to 0^+} \sup \sqrt{\phi(fgx_n, gy_n, gz_n)}. \]

Then by assumption (2.3.2), we have $\delta < 1$.

Let $k$ be such that $\delta < k < 1$. Then, there exists $n_0 \in N$ such that
\[ \sqrt{\phi(fgx_n, gy_n, gz_n)} < k, \text{ for each } n \geq n_0 \]

Thus from (2.7), we get
\[ f(gx_{n+1}, gy_{n+1}, gz_{n+1}) < k \cdot f(gx_n, gy_n, gz_n), \text{ for each } n \geq n_0. \]
Hence by induction, for each \( n \geq n_0 \), we have
\[
    f(gx_{n+1}, gy_{n+1}, gz_{n+1}) < k^{n+1-n_0} f(gx_{n_0}, gy_{n_0}, gz_{n_0}). \tag{2.9}
\]

Since \( \phi(t) \geq a > 0 \) for all \( t > 0 \), from (2.6), (2.9) and for each \( n \geq n_0 \), we have
\[
    d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}) < \frac{1}{\sqrt{k^n}} k^{n-n_0} f(gx_{n_0}, gy_{n_0}, gz_{n_0}). \tag{2.10}
\]

Now we consider for \( m > n \), we have
\[
    d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m) \\
    \leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\
    + d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2}) + \cdots + d(gy_{m-1}, gy_m) \\
    + d(gz_n, gz_{n+1}) + d(gz_{n+1}, gz_{n+2}) + \cdots + d(gz_{m-1}, gz_m) \\
    \leq \frac{1}{\sqrt{k^n}} k^{n-n_0} f(gx_{n_0}, gy_{n_0}, gz_{n_0}) + \frac{1}{\sqrt{k^n}} k^{n-1-n_0} f(gx_{n_0}, gy_{n_0}, gz_{n_0}) \\
    \quad + \cdots + \frac{1}{\sqrt{k^n}} k^{m-1-n_0} f(gx_{n_0}, gy_{n_0}, gz_{n_0}) \\
    \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

Hence \( \{gx_n\}, \{gy_n\} \) and \( \{gz_n\} \) are Cauchy sequences in \((X, d)\). Suppose \( g(X) \) is complete, there exist \( u, v, w \in g(X) \) such that
\[
    \lim_{n \to \infty} gx_n = u = go, \lim_{n \to \infty} gy_n = v = g\beta \text{ and } \lim_{n \to \infty} gz_n = w = g\gamma,
\]
for some \( \alpha, \beta, \gamma \in X \).

Now, we show that \((go, g\beta, g\gamma)\) is triple coincidence point of \( F \) and \( g \).

Since \( f \) is lower semi continuous from (2.8), we have
\[
    0 \leq f(go, g\beta, g\gamma) = D(go, F(\alpha, \beta, \gamma)) + D(g\beta, F(\beta, \alpha, \beta)) + D(g\gamma, F(\gamma, \beta, \alpha)) \\
    \leq \lim_{n \to \infty} \inf f(gx_n, gy_n, gz_n) = 0.
\]

Hence, we get
\[
    D(go, F(\alpha, \beta, \gamma)) = D(g\beta, F(\beta, \alpha, \beta)) = D(g\gamma, F(\gamma, \beta, \alpha)) = 0,
\]
which implies that
\[
    go \in F(\alpha, \beta, \gamma), \ g\beta \in F(\beta, \alpha, \beta) \text{ and } g\gamma \in F(\gamma, \beta, \alpha).
\]

Thus \((\alpha, \beta, \gamma)\) is triple coincidence point of \( F \) and \( g \).

**Example 2.4** Let \( X = [0, 1] \) and \( d : X \times X \to \mathbb{R} \) by \( d(x, y) = |x - y| \), then \((X, d)\) is complete metric space and we define \( \leq \) by
\[
    x \leq y \iff x \leq y
\]
then \( \leq \) is partial order relation.

We define \( g : X \to X \times X \times X \to CB(X) \) by \( F(x, y, z) = [x, 1] \), \( \forall x, y, z \in X \).

Then
\[
    f(gx, gy, gz) = D(gx, F(x, y, z)) + D(gy, F(y, x, y)) + D(gz, F(z, y, x)) \\
    = \inf \{ d(gx, a) : a \in [x, 1] \} + \inf \{ d(gy, b) : b \in [y, 1] \} + \inf \{ d(gz, c) : c \in [z, 1] \} \\
    = d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x)) \\
    = |\frac{1}{2} - x| + |\frac{1}{2} - y| + |\frac{1}{2} - z| \\
    = \frac{3}{2} - |\frac{1}{2} + \frac{1}{2} - x| - |\frac{1}{2} - y| - |\frac{1}{2} - z| \\
    = \frac{3}{2} - \frac{3}{2}.
\]

Also let \( \phi : [0, \infty) \to [a, 1], 0 \leq a < 1 \) by \( \phi(t) = \frac{t}{t+1} \) it is clear that \( \lim_{t \to t^+} \phi(t) < 1 \) for each \( t \in [0, +\infty) \).

Without loss generality we choose \( g0 = gu \leq gx, g0 = gv \leq gy \) and \( g0 = gw \leq gy \).

It is clear that
\[
    \sqrt{\phi(f(gx, gy, gz))} [d(gx, gu) + d(gy, gv) + d(gz, gw)] \leq f(gx, gy, gz)
\]
such that
\[
    f(gu, gv, gw) \leq \phi(f(gx, gy, gz)) [d(gx, gu) + d(gy, gv) + d(gz, gw)].
\]
Hence all conditions of Theorem 2.3 are satisfied and \((0, 0, 0)\) is the coincidence point of \( g \) and \( F \).
Corollary 2.5 Let $(X, d)$ be a metric space endowed with a partial order $\preceq$ and $\Delta \neq \emptyset$, that is there exist $(x_0, y_0, z_0) \in \Delta$. Suppose that $F : X \times X \times X \to CL(X)$ has a $\Delta$-property such that $f : X \times X \times X \to [0, +\infty)$ given by

$$f(x, y, z) := D(x, F(x, y, z)) + D(y, F(y, x, y)) + D(z, F(z, y, x)),$$

is lower semi-continuous and that there exists a function $\phi : [0, +\infty) \to [a, 1)$, $0 < a < 1$, satisfying $\lim_{t \to t^+} \sup \phi(t) < 1$, for each $t \in [0, +\infty)$. If for any $(x, y, z) \in \Delta$ there exist $u \in F(x, y, z)$, $v \in F(y, x, y)$ and $w \in F(z, y, x)$ satisfying

$$\sqrt{\phi(f(x, y, z))}[d(x, u) + d(y, v) + d(z, w)] \leq f(x, y, z)$$

such that

$$f(u, v, w) \leq \phi(f(x, y, z))[d(x, u) + d(y, v) + d(z, w)].$$

Then, $F$ has a triple fixed point.

**Proof.** If we take $g = I$ (identity map), then remaining proof follows from Theorem 2.3.

### 3 Discussion and Conclusions

The main Theorem 2.3 is an extension of Theorem of [6] from coupled coincidence point to tripled coincidence point. We have given an example to illustrate our main theorem. We also obtain a corollary from our Theorem 2.3.

### REFERENCES


