Deferred Statistical Cluster Points of Real Valued Sequences

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Abstract In this paper, the concept of deferred statistical cluster points of real valued sequences is defined and studied by using deferred density of the subset of natural numbers. For $p(n)$ and $q(n)$ satisfying certain conditions, we give some results for the set of deferred statistical cluster points $\Gamma_{D_{p,q}}(x)$. We provide some counter examples regarding $\Gamma_{D_{p,q}}(x)$. Also we obtain some inclusion results for $\Gamma_{D_{p,q}}(x)$. At last we consider the case $q(n) = \lambda(n)$ and $p(n) = \lambda(n-1)$ where the sequence $\lambda = \{\lambda(n)\}$ is strictly increasing sequence of positive natural numbers with $\lambda(0) = 0$.

Keywords Natural density, statistical cluster points, statistically convergent sequence

1 Introduction and notations

The concept of statistical convergence was introduced by H. Fast [8] and I.J. Steinhaus [23] independently in 1951. Since then, this subject was applied in different areas of mathematics such as in number theory by P. Erdős-G. Tenenbaum [7] and summability theory by A.R. Freedman-J.J. Sember-M. Raphael [9], etc.

Some properties of statistical convergence were studied by J. Conner in [3, 4], J.A. Fridy [10], J.A. Fridy-C. Orhan [12], T. Salat [21], I.J. Schenberg [22] and the others.

This subject is closely related to the subject of asymptotic (natural) density of the subset of natural numbers [2] and its root goes to A. Zygmund [25].

By using asymptotic density, the concept of statistical cluster points of real valued sequences was first introduced by J.A. Fridy [11]. Some generalizations of this concept have been studied by using regular summability methods in [5, 6, 13, 16, 20, 24].

In 1932, R.P. Agnew [1] defined the deferred Cesaro mean $D_{p,q}$ of the sequence $x = (x_k)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive nat-

ural numbers satisfying

$$p(n) < q(n) \text{ and } \lim_{n \rightarrow \infty} q(n) = \infty. \quad (1)$$

Let K be a subset of $\mathbb{N}$, and denote the set $\{k : p(n) < k \leq q(n), k \in K\}$ by $K_{p,q}(n)$. The deferred density of K is defined by

$$\delta_{p,q}(K) := \lim_{n \rightarrow \infty} \frac{1}{q(n) - p(n)} |K_{p,q}(n)| \quad (2)$$

whenever the limit exists. The vertical bars in (2) indicate the cardinality of the set $K_{p,q}(n)$.

Because of $\delta_{p,q}(K)$ does not exists for all $K \subset \mathbb{N}$, it is convenient to use upper deferred asymptotic density of K , defining by

$$\delta_{p,q}^*(K) = \limsup_{n \rightarrow \infty} \frac{|\{k : p(n) + 1 \leq k \leq q(n), k \in K\}|}{q(n) - p(n)}.$$

It is clear that,

- i) if $\delta_{p,q}(K)$ exists, then $\delta_{p,q}(K) = \delta_{p,q}^*(K)$,
- ii) $\delta_{p,q}(K) \neq 0$ if and only if $\delta_{p,q}^*(K) > 0$,
- iii) if $K \subset M$, then $\delta_{p,q}^*(K) \leq \delta_{p,q}^*(M)$

A real valued sequence $x = (x_n)$ is deferred statistical convergent to l , if the limit

$$\lim_{n \rightarrow \infty} \frac{1}{q(n) - p(n)} |\{p(n) < k \leq q(n) : |x_k - l| \geq \varepsilon\}| = 0, \quad (3)$$

exists for every $\varepsilon > 0$. (see [14, 15]).

It is clear that:

- iv) If $q(n) = n$, $p(n) = 0$ then (2) and (3) coincide the usual asymptotic density and statistical convergence respectively [10].
- v) If $q(n) = \lambda(n)$, $p(n) = n - \lambda(n)$ for the sequence $\lambda(n)$ satisfying $\lambda(n+1) \leq \lambda(n) + 1$ and $\lambda(1) = 1$, then (2) and (3) coincide S_λ -density and S_λ -convergence which was defined and studied by M. Mursaleen [17].

- vi) If $q(n) = \lambda(n)$, $p(n) = 0$ for the sequence $\lambda(n)$ such that it is strictly increasing sequence of natural numbers with $\lambda(0) = 0$, then (2) and (3) coincide SC_λ -density and SC_λ -convergence which was defined by [18].

Definition 1.1. The number γ is called deferred statistical cluster point of $x = (x_k)$, for every $p(n)$ and $q(n)$ satisfying (1), if for every $\varepsilon > 0$ the set

$$\{p(n) < k \leq q(n) : |x_k - \gamma| < \varepsilon\}$$

does not have deferred density zero i.e.,

$$\lim_{n \rightarrow \infty} \frac{|\{p(n) < k \leq q(n) : |x_k - \gamma| < \varepsilon\}|}{q(n) - p(n)} \neq 0 \quad (4)$$

and the set of deferred statistical cluster points of the sequence $x = (x_k)$ is denoted by $\Gamma_{D_{p,q}}(x)$, i.e.,

$$\Gamma_{D_{p,q}}(x) := \{\gamma : \gamma \text{ satisfies (4)}\}.$$

This definition is generalized version of the statistical cluster point definition given by J.A. Fridy in [11].

2 Main Results

2.1 Some properties of deferred statistical cluster points

In this section, some topological properties of deferred statistical cluster points of the real valued sequences are going to be investigated.

Theorem 2.1. If the sequence $x = (x_n)$ is deferred statistical convergence to l , then $\Gamma_{D_{p,q}}(x)$ contains only the elements l .

Proof. Assume that the sequence $x = (x_n)$ is deferred statistical convergence to l . Then, for every $\varepsilon > 0$, the limit relation

$$\lim_{n \rightarrow \infty} \frac{|\{k : p(n) + 1 \leq k \leq q(n), |x_k - l| \geq \varepsilon\}|}{q(n) - p(n)} = 0 \quad (5)$$

hold. It means that

$$\lim_{n \rightarrow \infty} \frac{|\{k : p(n) + 1 \leq k \leq q(n), |x_k - l| < \varepsilon\}|}{q(n) - p(n)} = 1 \neq 0.$$

Therefore, $l \in \Gamma_{D_{p,q}}(x)$. Now let us assume that the set $\Gamma_{D_{p,q}}(x)$ contains l' which different from l , i.e., $l \neq l'$. Take into consider $\varepsilon = \frac{1}{2}|l - l'|$. Since $x = (x_n)$ is deferred statistical convergence to l , then (5) is hold for this ε . It means that deferred asymptotic density of the elements $x = (x_n)$ belonging to the ε -neighborhood of l is 1. Consequently, the deferred asymptotic density of the elements of (x_n) belonging to the ε -neighborhood of l' is zero. That is,

$$\lim_{n \rightarrow \infty} \frac{|\{k : p(n) + 1 \leq k \leq q(n), |x_k - l'| < \varepsilon\}|}{q(n) - p(n)} = 0.$$

This is contradiction to assumption on l' . $\square$

Remark 2.1. The inverse of Theorem 2.1 is not true.

There exists a sequence such that the set of deferred statistical cluster points has unique elements but it is not deferred statistical convergence to this point. Let us consider the sequence $x = (x_n)$ where

$$x_n := \begin{cases} \frac{1}{n}, & n \text{ even}, \\ n, & n \text{ odd}. \end{cases}$$

It is clear that $0 \in \Gamma_{D_{0,n}}(x)$ but it is not deferred statistical convergence to zero.

Remark 2.2. Assume that the sequence $x = (x_n)$ is monotone increasing (decreasing). If $\sup x_n < \infty$ ($\inf x_n < \infty$), then $\sup x_n \in \Gamma_{D_{p,q}}(x)$, ($\inf x_n \in \Gamma_{D_{p,q}}(x)$) respectively.

Proof. Here we will give the proof for only the monotone increasing sequence. From the definition of supremum for any $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that the following inequality

$$\sup x_n - \varepsilon < x_{n_0} \leq \sup x_n$$

hold.

Since the sequence is monotone increasing, then we have

$$\sup x_n - \varepsilon < x_{n_0} < x_n \leq \sup x_n < \sup x_n + \varepsilon$$

for all $n > n_0$. It means that for any $\varepsilon > 0$ there exist a $n_0(\varepsilon) \in \mathbb{N}$ such that the inequality

$$|x_n - \sup x_n| < \varepsilon$$

holds for all $n > n_0$. From this discussion the following inclusion

$$\mathbb{N} - \{1, 2, \dots, n_0\} \subset \{n : |x_n - \sup x_n| < \varepsilon\}$$

holds. Since $\delta_{p,q}(\mathbb{N} - \{1, 2, \dots, n_0\}) = 1$, then $\delta_{p,q}(\{n : |x_n - \sup x_n| < \varepsilon\}) \neq 0$. This gives the desired proof. $\square$

Recall that the distance between $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ is

$$d(A, B) = \inf\{|a - b| : a \in A, b \in B\}$$

Theorem 2.2. Let $x = (x_n)$ be a real valued sequence. If $\Gamma_{D_{p,q}}(x) \neq \emptyset$, then $d(\Gamma_{D_{p,q}}(x), x) = 0$.

Proof. Assume that $\Gamma_{D_{p,q}}(x) \neq \emptyset$. Let us consider an arbitrary element $y \in \Gamma_{D_{p,q}}(x)$. Then, we have for an arbitrary positive ε ,

$$\lim_{n \rightarrow \infty} \frac{|\{k : p(n) + 1 \leq k \leq q(n), |x_k - y| < \varepsilon\}|}{q(n) - p(n)} \neq 0.$$

So, the set $A_\varepsilon := \{x_k : |x_k - y| < \varepsilon\}$ has at least countable elements of $x = (x_n)$ for an arbitrary positive ε . Therefore,

$$0 \leq d(\Gamma_{D_{p,q}}(x), x) = \inf\{|y - x_k| : k \in \mathbb{N}\} \leq \varepsilon$$

is hold. This gives the desired proof. $\square$

Theorem 2.3. If $\Gamma_{D_{p,q}}(x) \neq \emptyset$ for any $p(n)$ and $q(n)$, then $\Gamma_{D_{p,q}}(x)$ is closed.

Proof. Let us assume that $\Gamma_{D_{p,q}}(x) \neq \emptyset$ for any $p(n)$ and $q(n)$. It is enough to show that $\mathbb{R} \setminus \Gamma_{D_{p,q}}(x)$ is an open set. Let $y \in \mathbb{R} \setminus \Gamma_{D_{p,q}}(x)$ be an arbitrary point. Since $y \notin \Gamma_{D_{p,q}}(x)$, then there exists an $\varepsilon > 0$ such that

$$\delta_{p,q}(\{p(n) < k \leq q(n) : |x_k - y| < \varepsilon\}) = 0.$$

If we denote the open interval $(y - \varepsilon, y + \varepsilon)$ by A , then we have

$$\delta_{p,q}(\{p(n) < k \leq q(n) : x_k \in A\}) = 0.$$

If we choose $\varepsilon_y := \frac{1}{2} \inf \{|x_k - y| : x_k \in A\}$, then it is clear that $\varepsilon_y < \varepsilon$ and $(y - \varepsilon_y, y + \varepsilon_y) \subset \mathbb{R} \setminus \Gamma_{D_{p,q}}(x)$. It means that y is an arbitrary interior point of $\mathbb{R} \setminus \Gamma_{D_{p,q}}(x)$. Therefore $\mathbb{R} \setminus \Gamma_{D_{p,q}}(x)$ is an open set. $\square$

Theorem 2.4. Let $x = (x_n)$ be a real valued sequence and $\gamma \in \mathbb{R}$ be an arbitrary fixed point. If $d(\gamma, x) \neq 0$, then $\gamma \notin \Gamma_{D_{p,q}}(x)$ for any $p(n)$ and $q(n)$.

Proof. From the hypothesis we have

$$d(\gamma, x) := \inf \{|x_k - \gamma| : k \in \mathbb{N}\} = m > 0.$$

From this assumption the inequality

$$|x_k - \gamma| \geq m$$

hold for all $k \in \mathbb{N}$.

It means that the open interval $(\gamma - m, \gamma + m)$ has no elements of the sequence $x = (x_n)$. So, we have

$$\delta_{p,q}(\{p(n) < k \leq q(n) : x_k \in (\gamma - m, \gamma + m)\}) = 0 \quad (6)$$

therefore, if we choose an arbitrary $\varepsilon < m$ then the relation

$$\delta_{p,q}(\{p(n) < k \leq q(n) : |x_k - \gamma| < \varepsilon\}) = 0$$

hold. Otherwise it contradicts with (6) since the inclusion

$$\begin{aligned} \{p(n) < k \leq q(n) : |x_k - \gamma| < \varepsilon\} &\subset \\ &\subset \{p(n) < k \leq q(n) : |x_k - \gamma| < m\}. \end{aligned}$$

$\square$

Remark 2.3. If $d(\gamma, x) = 0$, it is not necessarily $\gamma \in \Gamma_{p,q}(x)$.

Let us consider the sequence $x = (x_n) = (\frac{1}{n})$ for all $n \in \mathbb{N}$. If we take $\gamma = \frac{1}{2}$, then $d(\frac{1}{2}, \frac{1}{n}) = 0$ but $\frac{1}{2} \notin \Gamma_{D_{p,q}}(x) = \{0\}$ when $q(n) = n$ and $p(n) = 0$.

Theorem 2.5. Let $x = (x_n)$ be a real valued sequence and $A \subset \mathbb{R}$ be an arbitrary set. If $d(A, x) \neq 0$, then

$$A \cap \Gamma_{D_{p,q}}(x) = \emptyset.$$

Proof. If the subset $A \subset \mathbb{R}$ is a singleton, then the proof is obtained from Theorem 2.4. Let $a^* \in A$ be an arbitrary element. There is $m > 0$ such that

$$|a^* - x_k| > m$$

since $d(A, x) > 0$. So, the intervals $(a^* - m, a^* + m)$ has no elements of $x = (x_n)$. Therefore, if we choose $\varepsilon < m$, the set $(a^* - m, a^* + m)$ contains no element of the sequence. Consequently, we have

$$\delta_{p,q}(\{p(n) < k \leq q(n) : |a^* - x_k| < \varepsilon\}) = 0$$

and $a^* \notin \Gamma_{D_{p,q}}(x)$.

$\square$

Remark 2.4. If $d(A, x) = 0$, it is not necessarily $A \cap \Gamma_{D_{p,q}}(x) \neq \emptyset$

Let us consider the sequence $x = (x_n) = (\frac{1}{n})$ for all $n \in \mathbb{N}$ and $A = (0, \infty)$. It clear that $A \cap \Gamma_{D_{p,q}}(x) = \emptyset$ but $d(A, x) = 0$.

2.2 Some inclusion results for $\Gamma_{D_{p,q}}(x)$

Throught this section, we consider the sequences of positive natural numbers $p(n), p'(n), q(n)$ and $q'(n)$

Denote the sets for only simply $E := \{p(n) : n \in \mathbb{N}\}$, $E' := \{p'(n) : n \in \mathbb{N}\}$, $F := \{q(n) : n \in \mathbb{N}\}$ and $F' := \{q'(n) : n \in \mathbb{N}\}$.

Theorem 2.6. If the set $F' \setminus F$ is finite and

$$\lim_{n \rightarrow \infty} \frac{q(n) - p(n)}{q'(n) - p(n)} = d \neq 0,$$

hold. Then, $\delta_{p,q'}(K) \neq 0$ implies $\delta_{p,q}(K) \neq 0$ for every $K \subseteq \mathbb{N}$.

Proof. Since the set $F' \setminus F$ is finite, then there exists a positive natural number N such that

$$\{q'(n) : n \geq N\} \subset \{q(n) : n \in \mathbb{N}\}.$$

For $n \geq N$ let $j(n)$ be a strictly increasing sequence such that $q'(n) := q(j(n))$. If $\delta_{p,q'}(K) \neq 0$ then the relation

$$\delta_{p,q'}^*(K) = \limsup_{n \rightarrow \infty} \frac{|\{p(n) + 1 \leq k \leq q'(n) : k \in K\}|}{q'(n) - p(n)} > 0$$

holds. So, we have

$$\begin{aligned} \frac{|\{p(n) + 1 \leq k \leq q(j(n)) : k \in K\}|}{q(j(n)) - p(n)} &\leq \\ &\leq \frac{q(n) - p(n)}{q(j(n)) - p(n)} \cdot \frac{|\{p(n) + 1 \leq k \leq q(n) : k \in K\}|}{q(n) - p(n)} \end{aligned}$$

and

$$\delta_{p,q}^*(K) = \limsup_{n \rightarrow \infty} \frac{|\{p(n) + 1 \leq k \leq q(n) : k \in K\}|}{q(n) - p(n)} > 0.$$

Hence $\delta_{p,q}(K) \neq 0$ and the proof is obtained. $\square$

Corollary 2.1. Let us assume that

$$\lim_{n \rightarrow \infty} \frac{q(n) - p(n)}{q'(n) - p(n)} = d \neq 0,$$

hold. Then, the followings are true:

i) If $F' \setminus F$ is finite, then

$$\Gamma_{D_{p,q}}(x) \supset \Gamma_{D_{p,q'}}(x),$$

ii) If $F' \Delta F$ is finite, then

$$\Gamma_{D_{p,q}}(x) = \Gamma_{D_{p,q'}}(x).$$

Theorem 2.7. If $E' \setminus E$ is finite and

$$\lim_{n \rightarrow \infty} \frac{q(n) - p'(n)}{q(n) - p(n)} = d \neq 0,$$

hold. Then, $\delta_{D_{p,q}}(K) \neq 0$ implies $\delta_{D_{p',q}}(K) \neq 0$ for every $K \subseteq \mathbb{N}$.

Proof. If $E' \setminus E$ is finite, then there exists a positive natural number N such that

$$\{p'(n) : n \geq N\} \subset \{p(n) : n \in \mathbb{N}\}$$

hold.

For $n \geq N$ let $j(n)$ be monotone increasing such that $p'(n) = p(j(n))$. If $\delta_{D_{p,q}}(K) \neq 0$ then

$$\delta_{D_{p,q}}^*(K) = \limsup_{n \rightarrow \infty} \frac{|\{p(n) + 1 \leq k \leq q(n) : k \in K\}|}{q(n) - p(n)} > 0.$$

From this we have

$$\begin{aligned} & \frac{|\{p(n) + 1 \leq k \leq q(n) : k \in K\}|}{q(n) - p(n)} \leq \\ & \leq \frac{q(n) - p(j(n))}{q(n) - p(n)} \cdot \frac{|\{p(j(n)) + 1 \leq k \leq q(n) : k \in K\}|}{q(n) - p(j(n))} \end{aligned}$$

and

$$\delta_{D_{p,q}}^*(K) = \limsup_{n \rightarrow \infty} \frac{|\{p'(n) + 1 \leq k \leq q(n) : k \in K\}|}{q(n) - p'(n)} > 0.$$

It gives $\delta_{D_{p',q}}(K) \neq 0$ and we obtained desired result. $\square$

Corollary 2.2. *Under the assumption of Theorem 2.7 the following statements are true:*

i) *If $E' \setminus E$ is finite, then*

$$\Gamma_{D_{p',q}}(x) \supset \Gamma_{D_{p,q}}(x),$$

ii) *If $E' \triangle E$ is finite, then*

$$\Gamma_{D_{p',q}}(x) = \Gamma_{D_{p,q}}(x).$$

Theorem 2.8. *Let us assume that*

$$p(n) \leq p'(n) < q'(n) \leq q(n)$$

and

$$\lim_{n \rightarrow \infty} \frac{q(n) - p(n)}{q'(n) - p'(n)} = d \neq 0$$

hold. Then, $\delta_{D_{p',q'}}(K) \neq 0$ implies $\delta_{D_{p,q}}(K) \neq 0$ for every $K \subseteq \mathbb{N}$.

Proof. If $\delta_{D_{p',q'}}(K) \neq 0$, then we have

$$\begin{aligned} \delta_{D_{p',q'}}^*(K) &= \\ &= \limsup_{n \rightarrow \infty} \frac{|\{p'(n) + 1 \leq k \leq q'(n) : k \in K\}|}{q'(n) - p'(n)} > 0, \end{aligned}$$

and the relation

$$\begin{aligned} & \frac{|\{p'(n) + 1 \leq k \leq q'(n) : k \in K\}|}{q'(n) - p'(n)} \leq \\ & \leq \frac{q(n) - p(n)}{q'(n) - p'(n)} \cdot \frac{|\{p(n) + 1 \leq k \leq q(n) : k \in K\}|}{q(n) - p(n)} \end{aligned}$$

hold. Therefore,

$$\delta_{D_{p,q}}^*(K) = \limsup_{n \rightarrow \infty} \frac{|\{p(n) + 1 \leq k \leq q(n) : k \in K\}|}{q(n) - p(n)} > 0.$$

Hence $\delta_{D_{p,q}}(K) \neq 0$, and the proof is ended. $\square$

Corollary 2.3. *Under the assumptions of Theorem 2.8, we have*

$$\Gamma_{D_{p,q}}(x) \supset \Gamma_{D_{p',q'}}(x).$$

Theorem 2.9. *Let $p = p(n)$ be an arbitrary sequence, $q(n) \leq n$ for all $n \in \mathbb{N}$ and*

$$\lim_{n \rightarrow \infty} \frac{n}{q(n) - p(n)} = d \neq 0 \quad (7)$$

hold. Then, $\delta_{D_{p,q}}(K) \neq 0$ implies $\delta(K) \neq 0$ for every $K \subseteq \mathbb{N}$.

Proof. If $\delta_{D_{p,q}}(K) \neq 0$ then,

$$\delta_{D_{p,q}}^*(K) = \limsup_{n \rightarrow \infty} \frac{|\{p(n) + 1 \leq k \leq q(n) : k \in K\}|}{q(n) - p(n)} > 0,$$

and the relation

$$\begin{aligned} & \frac{|\{p(n) + 1 \leq k \leq q(n) : k \in K\}|}{q(n) - p(n)} \leq \\ & \frac{n}{q(n) - p(n)} \cdot \frac{|\{k \leq n : k \in K\}|}{n} \end{aligned}$$

hold. Therefore

$$\delta^*(K) = \limsup_{n \rightarrow \infty} \frac{|\{k \leq n : k \in K\}|}{n} > 0$$

Hence $\delta(K) \neq 0$. End of proof. $\square$

Let us note that if $q(n) = n$ for all $n \in \mathbb{N}$, the condition (7) is omitted in the Theorem 2.9.

Corollary 2.4. *Under the conditions of Theorem 2.9 the inclusion*

$$\Gamma(x) \supset \Gamma_{D_{p,q}}(x)$$

hold.

2.3 Some inclusion results for $\Gamma_{D_\lambda}(x)$

In this section we consider the case $q(n) := \lambda(n)$ and $p(n) := \lambda(n-1)$ when the sequence $\lambda = \{\lambda(n)\}_{n \in \mathbb{N}}$ is a strictly increasing sequence of positive natural numbers and $\lambda(0) = 0$.

Theorem 2.10. *If the limit $\lim_{n \rightarrow \infty} \frac{\lambda(n)}{\lambda(n) - \lambda(n-1)} = d \neq 0$ is hold. Then, $\delta_{D_\lambda}(K) \neq 0$ implies $\delta_{CS_\lambda}(K) \neq 0$ for every subset K of $\mathbb{N}$.*

Proof. If $\delta_{D_\lambda}(K) \neq 0$ then we have

$$\delta_{D_\lambda}^*(K) = \limsup_{n \rightarrow \infty} \frac{|\{k : \lambda(n-1) + 1 \leq k \leq \lambda(n), k \in K\}|}{\lambda(n) - \lambda(n-1)} > 0,$$

and the following inequality

$$\begin{aligned} & \frac{1}{\lambda(n) - \lambda(n-1)} |\{k : \lambda(n-1) + 1 \leq k \leq \lambda(n), k \in K\}| \leq \\ & \frac{\lambda(n)}{\lambda(n) - \lambda(n-1)} \frac{1}{\lambda(n)} |\{k : 1 \leq k \leq \lambda(n), k \in K\}| \end{aligned}$$

hold. Therefore, under the assumption we have

$$\delta_{CS_\lambda}^*(K) := \limsup_{n \rightarrow \infty} \frac{1}{\lambda(n)} |\{k : 1 \leq k \leq \lambda(n), k \in K\}| > 0.$$

So, $\delta_{CS_\lambda}(K) \neq 0$. $\square$

Corollary 2.5. *Under the condition of Theorem 2.10, the inclusion*

$$\Gamma_{CS_\lambda}(x) \supset \Gamma_{D_\lambda}(x)$$

hold.

Theorem 2.11. *Let $G = \{\lambda(n)\}_{n \in \mathbb{N}}$ and $G' = \{\lambda'(n)\}_{n \in \mathbb{N}}$ be an infinite subset of positive natural numbers. If $G' - G$ is finite set and the limit*

$$\lim_{n \rightarrow \infty} \frac{\lambda'(n) - \lambda'(n-1)}{\lambda(n) - \lambda(n-1)} = d \neq 0,$$

hold. Then, $\delta_{D_\lambda}(K) \neq 0$ implies $\delta_{D_{\lambda'}}(K) \neq 0$ for every $K \subseteq \mathbb{N}$.

Proof. If the set $G' - G$ is finite, then there exists a positive natural number N such that the inclusion

$$\{\lambda'(n) : n \geq N\} \subset G$$

hold. It means that there exists a monotone increasing sequence $(j(n))_{n \in \mathbb{N}}$ tending infinity such that $\lambda'(n) = \lambda(j(n))$.

If $\delta_{D_\lambda}(K) \neq 0$, then we have

$$\begin{aligned} \delta_{D_\lambda}^*(K) &= \\ &= \limsup_{n \rightarrow \infty} \frac{|\{k : \lambda(n-1) + 1 \leq k \leq \lambda(n), k \in K\}|}{\lambda(n) - \lambda(n-1)} > 0. \end{aligned}$$

Also, the inequality

$$\begin{aligned} &\frac{|\{k : \lambda(n-1) + 1 \leq k \leq \lambda(n), k \in K\}|}{\lambda(n) - \lambda(n-1)} \leq \\ &\leq \frac{\lambda(j(n)) - \lambda(j(n)-1)}{\lambda(n) - \lambda(n-1)} \times \\ &\times \frac{|\{k : \lambda(j(n)-1) + 1 \leq k \leq \lambda(j(n)), k \in K\}|}{\lambda(j(n)) - \lambda(j(n)-1)} \end{aligned}$$

hold. After taking upper limit we have

$$\limsup_{n \rightarrow \infty} \frac{|\{k : \lambda(j(n)-1) + 1 \leq k \leq \lambda(j(n)), k \in K\}|}{\lambda(j(n)) - \lambda(j(n)-1)} > 0.$$

Therefore,

$$\limsup_{n \rightarrow \infty} \frac{|\{k : \lambda'(n-1) + 1 \leq k \leq \lambda'(n), k \in K\}|}{\lambda'(n) - \lambda'(n-1)} > 0$$

and this gives $\delta_{D_{\lambda'}}(K) \neq 0$. This finishes the proof. $\square$

Corollary 2.6. *Let us assume that $\lim_{n \rightarrow \infty} \frac{\lambda'(n) - \lambda'(n-1)}{\lambda(n) - \lambda(n-1)} = d \neq 0$. Then, the following statements are true:*

i) *If $G' - G$ is finite, then*

$$\Gamma_{D_\lambda}(x) \subset \Gamma_{D_{\lambda'}}(x)$$

holds,

ii) *If $G' \Delta G$ is finite, then*

$$\Gamma_{D_\lambda}(x) = \Gamma_{D_{\lambda'}}(x)$$

holds.

REFERENCES

- [1] R.P. AGNEW, *On deferred Cesaro Mean*, Ann. of Math., **33** (1932), 413–421.
- [2] R.P. BUCK, *Generalized asymptotic density*, Amer. J. Math. Comm., **75** (1953), 335–346.
- [3] J.S. CONNOR, *The statistical and strong p-Cesaro convergence of sequences*, Analysis, **8** (1988) 47–63.
- [4] J.S. CONNOR, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., **32** (1989), 194–198.
- [5] K. DEMIRCI, *A-statistical core of a sequence*, Math., **33** No2 (2000), 343–353.
- [6] K. DEMIRCI, *On A-statistical cluster points*, Glasnik Math., Vol37 **57** (2002), 293–301.
- [7] P. ERDÖS, G. TENENBAUM, *Sur les densites de certaines suites d'entiers*, Proceeding of the London Math. Soc. vol 59, **3** (1989), 417–438.
- [8] H. FAST, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244.
- [9] A.R. FREEDMAN, J.J. SEMBER, P. APHAEL, *Some Cesaro type summability Spaces*, Proc. London Math. Soc., **37** (1978), 301–313.
- [10] J.A. FRIDY, *On statistical convergence*, Analysis **5** (1985), 301–313.
- [11] J.A. FRIDY, *Statistical limit points*, Proc. of the American Math. Soc. Vol118, number4, August1993, 1187–1191.
- [12] J.A. FRIDY, C. ORHAN, *Statistical limit superior and statistical limit inferior*, Proc. of the American Math. Soc. Vol125, number12, December1997, 3625–3631.
- [13] P. KOSTYRKO, M. MACAJ, T. SALAT, O. STROUCH, *On statistical limit points*, Proc. Amer. Math. Soc. **vol129**, number9, (2000), 2647–2654.
- [14] M. KUCUKASLAN, M. YILMAZTURK, *Deferred statistical convergence*, Kyunpook Mathematical Journal, Submitted, (2013).
- [15] M. KUCUKASLAN, M. YILMAZTURK, *On strongly Cesaro Summability and deferred statistical convergence of sequences*, BEÜ Journal of Science and Technology, **3**, (2013), 22–25.
- [16] M. MAMEDOV, S. PELIVAN, *Statistical cluster points and Turnpike Theorem in Nonconvex Problems*, Journal of Mathematical Analysis and Applications **256**, (2001), 686–693.
- [17] M. MURSALEEN, *λ -statistical convergence*, Mathematica Slovaca, **vol 50.1** (2000), 111–115.
- [18] J.A. OSIKIEWICZ, *Summability of matrix submethods and Spliced Sequences*, Ph.D. Thesis, Kent State University, 1997.
- [19] S. PEHLIVAN, M. GÜRDAL, B. FISHER, *Lacunary statistical cluster points of sequences*, Mathematical communications, **11** (2006), 39–46.
- [20] S. PEHLIVAN, H. ALBAYRAK, H.Z. TOYGANÖZÜ, *The Theory of Convergence and the set of statistical Cluster Points*, Advances in Dynamic System and Applications, **vol6**, number1, (2011), 111–119.

- [21] T. SALAT, *On statistically convergent sequences of real numbers*, Math. Slovaca **30**(1980), 139–150.
- [22] I.J. SCHOENBERG, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66**, (1959), 361–375.
- [23] H. STEINHAUS, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1951), 73–74.
- [24] J. ZEAGER, *Statistical Limit Point Theorems*, Internat. J. Math. and Math. Sci., **vol23**, No11,(2000),741–752.
- [25] A. ZYGMUND, *Trigonometric Series*, Cambridge Univ. Press U.K. (1979).