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Goursat Problem in the Non-Classical Treatment for a Sixth Order Pseudoparabolic Equation

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Abstract In this paper substantiated for a differential equation of pseudoparabolic type with discontinuous coefficients a Goursat problem with non-classical boundary conditions is considered, which requires no matching conditions. Equivalence of these conditions boundary condition is substantiated classical, in the case if the solution of the problem in the anisotropic S. L. Sobolev's space is found. The considered equation as a pseudoparabolic equation generalizes not only classic equations of mathematical physics (heat-conductivity equations, string vibration equation) and also many models differential equations (telegraph equation, Aller's equation , moisture transfer generalized equation, Manjeron equation, Boussinesq - Love equation and etc.). It is grounded that the Goursat boundary conditions in the classic and non-classic treatment are equivalent to each other, and such boundary conditions are demonstrated in geometric form. Even from geometric interpretation can see that the grounded non-classic treatment doesn't require any additional conditions of agreement type. Thus, namely in this paper, the non-classic problem with Goursat conditions is grounded for a pseudoparabolic equation of sixth order. For simplicity, this was demonstrated for one model case in one of S.L. Sobolev anisotropic space $W_p^{(2,4)}(G)$.

Keywords Goursat problem, pseudoparabolic equation, equation with discontinuous coefficients

1. Introduction

Pseudoparabolic equations are attracted for sufficiently adequate description of a great deal of real processes occurring in the nature, engineering and etc. In particular, many processes arising in the theory of fluid filtration in cracked media are described by discontinuous coefficient pseudoparabolic equations .

Urgency of investigations conducted in this field is explained by appearance of local and non-local problems for discontinuous coefficients equations connected with different applied problems. Such type problems arise for example, while studying the problems of moisture, transfer in soils, heat transfer in heterogeneous media, diffusion of thermal neutrons in inhibitors, simulation of different biological processes, phenomena and etc.

In the present paper, here consider Goursat problem for sixth order equation with discontinuous coefficients. The coefficients in this pseudoparabolic equation are not necessarily differentiable; therefore, there does not exist a formally adjoint differential equation making a certain sense. For this reason, this question cannot be investigated by the well-known methods using classical integration by parts and Riemann functions or classical-type fundamental solutions. The theme of the present paper, devoted to the investigation Goursat problem for sixth order differential equations of pseudoparabolic type, according to the above-stated is very actual for the solution of theoretical and practical problems. From this point of view, the paper is devoted to the actual problems of mathematical physics and computational mathematics.

2. Problem Statement

Consider equation

$$\left(V_{2,4}u\right)(x) \equiv D_1^2 D_2^4 u(x) + \sum_{\substack{i=0 \\ i+j<6}}^{2} \sum_{j=0}^{4} a_{i,j}(x) D_1^i D_2^j u(x) = Z_{2,4}(x) \in L_p(G) \quad (1)$$

Here $u(x) \equiv u(x_1, x_2)$ is a desired function determined on $G; a_{i,j}(x)$ are the given measurable functions on $G = G_1 \times G_2$, where $G_k = (0, h_k),\ k = \overline{1,2};\ Z_{2,4}(x)$ is a given measurable function on G ; $D_k^\xi = \partial^\xi / \partial x_k^\xi$ is a generalized differentiation operator in S.L.Sobolev sense, D_k^0 is an identity transformation operator.

Equation (1) is a hyperbolic equation and has two characteristics $x_1 = const, x_2 = const$, one of which is double-fold the second one is four-fold. Therefore, in some sense we can consider equation (1) as a pseudoparabolic equation [1]. This equation is a Boussinesq - Love generalized equation from the vibrations theory [2] and Aller's equation under mathematical modeling [3, p.261] of the moisture absorption process in biology.

In the present paper equation (1) is considered in the general case when the coefficients $a_{i,j}(x)$ are non-smooth functions satisfying only the following conditions:

$$a_{i,j}(x) \in L_p(G),\ i=\overline{0,1}\ \ j=\overline{0,3};$$

$$a_{2,j}(x) \in L_{\infty,p}^{x_1,x_2}(G),\ j=\overline{0,3};$$

$$a_{i,4}(x) \in L_{p,\infty}^{x_1,x_2}(G),\ i=\overline{0,1}$$

Under these conditions, we'll look for the solution $u(x)$ of equation (1) in S.L.Sobolev anisotropic space $W_p^{(2,4)}(G) \equiv \left\{u(x): D_1^i D_2^j u(x) \in L_p(G),\ i=\overline{0,2},\ j=\overline{0,4}\right\}$ where $1 \le p \le \infty$. We'll define the norm in the space $W_p^{(2,4)}(G)$ by the equality

$$\|u\|_{W_p^{(2,4)}(G)} \equiv \sum_{i=0}^{2}\sum_{j=0}^{4}\left\|D_1^i D_2^j u\right\|_{L_p(G)}$$

For equation (1) we can give the classic form Goursat condition in the form (see. Fig. 1):

$$\begin{cases} u(0,x_2)=\varphi_1(x_2);\quad u(x_1,0)=\psi_1(x_1); \\ \left.\dfrac{\partial u(x)}{\partial x_1}\right|_{x_1=0}=\varphi_2(x_2);\ \left.\dfrac{\partial u(x)}{\partial x_2}\right|_{x_2=0}=\psi_2(x_1); \\ \left.\dfrac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=0}=\psi_3(x_1);\ \left.\dfrac{\partial^3 u(x)}{\partial x_2^3}\right|_{x_2=0}=\psi_4(x_1), \end{cases} \tag{2}$$

where $\varphi_1(x_2), \varphi_2(x_2)$ and $\psi_k(x_1),\ k=\overline{1,4}$ are the given measurable functions on G. It is obvious that in the case of conditions (2), in addition to the conditions

$$\varphi_1(x_2) \in W_p^{(4)}(G_2),\ \varphi_2(x_2) \in W_p^{(4)}(G_2)$$

and

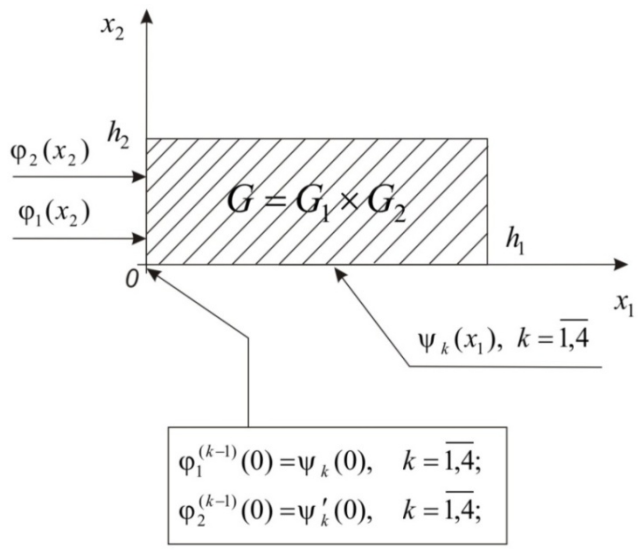
$$\psi_k(x_1) \in W_p^{(2)}(G_1),\ k=\overline{1,4}$$

the given functions should also satisfy the following agreement conditions:

$$\begin{cases} \varphi_1(0)=\psi_1(0);\ \ \varphi_2(0)=\psi_1'(0) \\ \varphi_1'(0)=\psi_2(0);\ \ \varphi_2'(0)=\psi_2'(0) \\ \varphi_1''(0)=\psi_3(0);\ \ \varphi_2''(0)=\psi_3'(0) \\ \varphi_1'''(0)=\psi_4(0);\ \ \varphi_2'''(0)=\psi_4'(0). \end{cases} \tag{3}$$

Consider the following non-classical initial-boundary conditions (see. Fig. 2) :

$$\begin{cases} V_{i,j}u \equiv D_1^i D_2^j u(0) = Z_{i,j} \in R, i=\overline{0,1}, j=\overline{0,3}; \\ (V_{2,j}u)(x_1) \equiv D_1^2 D_2^j u(x_1,0) = Z_{2,j}(x_1) \in L_p(G_1), j=\overline{0,3}; \\ (V_{i,4}u)(x_2) \equiv D_1^i D_2^4 u(0,x_2) = Z_{i,4}(x_2) \in L_p(G_2), i=\overline{0,1}; \end{cases} \tag{4}$$



$$x=(x_1,x_2)\in G,\ G=G_1\times G_2,\ \ G_k=(0,h_k),\ k=\overline{1,2}$$

Figure 1. Geometric interpretation of Goursat classical boundary conditions.

3. Methodology

Therewith, the important principal moment is that the considered equation possesses discontinuous coefficients satisfying only some p -integrability and boundedness conditions i.e. the considered pseudoparabolic operator $V_{2,4}$ has no traditional conjugated operator. In other words, the Riemann function for this equation can't be investigated by the classical method of characteristics. In the papers [4-5] The Riemann function is determined as the solution of an integral equation. This is more natural than the classical way for deriving the Riemann function. The matter is that in the classic variant, for determining the Riemann function, the rigid smooth conditions on the coefficients of the equation are required.

The Riemann's method does not work for differential equations with discontinuous coefficients.

In the present paper, a method that essentially uses modern methods of the theory of functions and functional analysis is worked out for investigations of such problems. In the main, this method it requested in conformity to pseudoparabolic equations of sixth order with triple characteristics. Notice that, in this paper the considered equation is a generation of many model equations of some processes (for example, heat-conductivity equations , telegraph equation, Aller's equation , moisture transfer generalized equation, Manjeron equation, equation, string vibrations equations and etc).

If the function $u \in W_p^{(2,4)}(G)$ is a solution of the classical form Goursat problem (1), (2), then it is also a solution of problem (1), (4) for $Z_{i,j}$, defined by the following equalities:

$$Z_{0,0} = \psi_1(0) = \varphi_1(0); \quad Z_{1,0} = \varphi_2(0) = \psi_1'(0);$$

$$Z_{0,1} = \psi_2(0) = \varphi_1'(0); \quad Z_{1,1} = \varphi_2'(0) = \psi_2'(0);$$

$$Z_{0,2} = \psi_3(0) = \varphi_1''(0); \quad Z_{1,2} = \varphi_2''(0) = \psi_3'(0);$$

$$Z_{0,3} = \psi_4(0) = \varphi_1'''(0) \quad Z_{1,3} = \varphi_2'''(0) = \psi_4'(0);$$

$$Z_{2,0}(x_1) = \psi_1''(x_1); \qquad Z_{2,1}(x_1) = \psi_2''(x_1)$$

$$Z_{2,2}(x_1) = \psi_3''(x_1); \quad Z_{2,3}(x_1) = \psi_4''(x_1);$$

$$Z_{0,4}(x_2) = \varphi_1^{(IV)}(x_2); \quad Z_{1,4}(x_2) = \varphi_2^{(IV)}(x_2)$$

The inverse one is easily proved. In other words, if the function $u \in W_p^{(2,4)}(G)$ is a solution of problem (1), (4), then it is also a solution of problem (1), (2) for the following functions:

$$\phi_1(x_2) = Z_{0,0} + x_2 Z_{0,1} + \frac{x_2^2}{2!} Z_{0,2} + \frac{x_2^3}{3!} Z_{0,3} + \int_0^{x_2} \frac{(x_2-\tau)^3}{3!} Z_{0,4}(\tau)\, d\tau; \quad (5)$$

$$\phi_2(x_2) = Z_{1,0} + x_2 Z_{1,1} + \frac{x_2^2}{2!} Z_{1,2} + \frac{x_2^3}{3!} Z_{1,3} + \int_0^{x_2} \frac{(x_2-\xi)^3}{3!} Z_{1,4}(\xi)\, d\xi; \quad (6)$$

$$\psi_1(x_1) = Z_{0,0} + x_1 Z_{1,0} + \int_0^{x_1} (x_1-\eta) Z_{2,0}(\eta)\, d\eta; \quad (7)$$

$$\psi_2(x_1) = Z_{0,1} + x_1 Z_{1,1} + \int_0^{x_1} (x_1-\nu) Z_{2,1}(\nu)\, d\nu; \quad (8)$$

$$\psi_3(x_1) = Z_{0,2} + x_1 Z_{1,2} + \int_0^{x_1} (x_1-\mu) Z_{2,2}(\mu)\, d\mu; \quad (9)$$

$$\psi_4(x_1) = Z_{0,3} + x_1 Z_{1,3} + \int_0^{x_1} (x_1-\tau) Z_{2,3}(\tau)\, d\tau; \quad (10)$$

Note that the functions (5)-(10) possess one important property, more exactly, for all $Z_{i,j}$, the agreement conditions (3) possessing the above-mentioned properties are fulfilled for them automatically. Therefore, equalities (5)-(10) may be considered as a general kind of all the functions $\varphi_1(x_2), \varphi_2(x_2)$ and $\psi_k(x_1)$ $k = \overline{1,4}$ satisfying the agreement conditions (3).

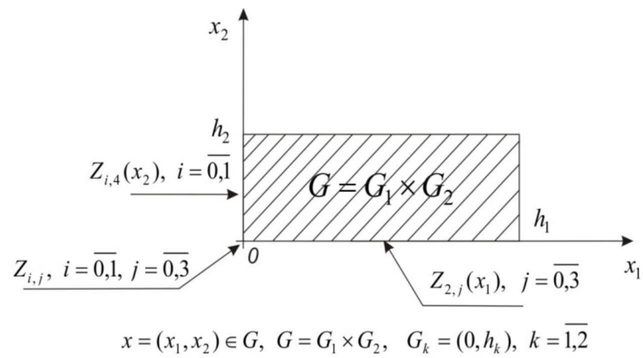


Figure 2. Geometrical interpretation of Goursat boundary conditions in non – classical statement

4. Result

So, the classical form Goursat problems (1), (2) and in non-classical treatment (1), (4) are equivalent in the general case. However, the Goursat problem in non-classical statement (1), (4) is more natural by statement than problem (1), (2). This is connected with the fact that in statement of problem (1), (4) the right sides of boundary conditions don't require additional conditions of agreement type. Note that some Goursat problems in non-classical treatments for hyperbolic and also pseudoparabolic equations were investigated in the author's papers [6-9].

5. Discussion and Conclusions

In this paper a non-classical type Goursat problem is substantiated for a pseudoparabolic equation with non-smooth coefficients and with a sixth order dominating derivative. Classic Goursat conditions are reduced to non-classic Goursat conditions by means of integral representations. Such statement of the problem has several advantages:

1) No additional agreement conditions are required in this statement;

2) One can consider this statement as a Goursat problem formulated in terms of traces in the S.L. Sobolev anisotropic

space $W_p^{(2,4)}(G)$;

3) In this statement the considered equation is a generalization of many model equations of some processes (e.g. heat-conductivity equations , telegraph equation, Aller's equation , moisture transfer generalized equation, Manjeron equation, Boussinesq - Love equation, string vibrations equations and etc.).

REFERENCES

[1] D.Colton, "Pseudoparabolic equations in one space variable", J. Different. equations, 1972, vol.12, No3, pp.559-565.

[2] A.P.Soldatov, M.Kh.Shkhanukov, "Boundary value problems with A.A.Samarsky general nonlocal condition for higher order pseudoparabolic equations", Dokl. AN SSSR, 1987, vol.297, No 3. pp.547-552 .

[3] A.M.Nakhushev, Equations of mathematical biology. M.: Visshaya Shkola, 1995, 301p.

[4] S.S.Akhiev, "Fundamental solution to some local and non - local boundary value problems and their representations ", DAN SSSR, 1983, vol.271, No 2, pp.265-269.

[5] V.I.Zhegalov, E.A.Utkina, "On a third order pseudoparabolic equation", Izv. Vuzov, Matem., 1999, No 10, pp.73-76.

[6] I.G.Mamedov, "On a new type three-dimensional Goursat problem for a hyperbolic equation with discontinuous coefficients", Mathematical simulation and boundary value problems, 2009, No 3, pp. 158-160.

[7] I.G.Mamedov, "A non-classical formula for integration by parts related to Gourrsat problem for a pseudoparabolic equation", Vladikavkazsky Matematicheskiy Zhurnal ,2011, vol.13, No 4, pp.40-51.

[8] I.G.Mamedov, "Goursat non - classic three dimensional problem for a hyperbolic equation with discontinuous coefficients", Vestnik Samarskogo Gosudarstvennogo Tekhnicheskogo Universiteta, 2010, No 1 (20), pp. 209-213.

[9] I.G.Mamedov, "Fundamental solution of initial boundary value problem for a fourth order pseudoparabolic equation with non-smooth coefficients", Vladikavkazsky Matematicheskiy Zhurnal, 2010, vol. 12, No 1, pp. 17-32.