Inequalities for the $s$th Derivative of A Polynomial with Prescribed Zeros

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Abstract Let $p(z)$ be a polynomial of degree $n$ which does not vanish in $|z| < k$, $k \geq 1$, then for $1 \leq R \leq k$ Bidkham and Dewan [J. Math. Anal. Appl. 166 (1992), 191–193] proved

$$\max_{|z|=R} |p'(z)| \geq \frac{n(R + k)^{n-1}}{(1 + k)^n} \max_{|z|=1} |p(z)|.$$  

In this paper we shall present several interesting generalizations and a refinement of this result which includes some results due to Malik, Govil and others. We also present a refinement of some other results.

Keywords Derivative of a polynomial, Zeros, Inequalities


1 Introduction

For $p \in p_n$ where $p_n$ be class of polynomials $p(z)$ of degree atmost $n$, we define

$$\|p\|_{\gamma} := \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |p(e^{i\theta})|^\gamma \right\}^\frac{1}{\gamma}, \quad 1 \leq \gamma < \infty$$

$$\|p\|_{\infty} := \max_{|z|=1} |p(z)| \quad \text{and} \quad m := \min_{|z|=k} |p(z)|.$$  

Let $p(z)$ be a polynomial of degree $n$, then according to a famous result known as Bernstein inequality (for reference see [13] or [12])

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.$$  

(1.1)

The result is best possible and equality holds for the polynomial having all its zeros at origin.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequality (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.$$  

(1.2)

As an extension of (1.2) Malik [8] verified that if $p(z)$ does not vanish in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |p(z)|.$$  

(1.3)

Equality holds for $p(z) = (z + k)^n$.

Bidkham and Dewan [2] obtained a generalization of (1.3) for the same class of polynomials by proving

$$\max_{|z|=R} |p'(z)| \geq \frac{n(R + k)^{n-1}}{1 + k} \max_{|z|=1} |p(z)|.$$  

(1.4)

The result is best possible and equality holds for $p(z) = (z + k)^n$.

In the reverse direction it was proved by Turan [9] that if $p(z)$ does not vanish in $|z| > 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|.$$  

(1.5)

Inequality (1.5) was refined by Aziz and Dawood [1] by showing that under the same hypothesis that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)| \right\}.$$  

(1.6)

Both the inequalities (1.5) and (1.6) are sharp and equality holds for $p(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. As an extension of (1.5), Govil [3] proved that if $p(z)$ is a polynomial of degree $n$ having all its zero in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1 + k} \max_{|z|=1} |p(z)|.$$  

(1.7)

Equality holds for $p(z) = (z + k)^n$, $k \leq 1$.

Whereas if $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$ with $s$-fold zeros at the origin, then Aziz and Shah [4] proved that

$$\max_{|z|=1} |p'(z)| \geq \frac{n + sk}{1 + k} \max_{|z|=1} |p(z)|.$$  

(1.8)

The result is sharp and extremal polynomial is $p(z) = z^s(z + k)^{n-s}$, $0 < s \leq n$. 


Govi and Dwatone [6] generalized inequality (1.6) of Aziz and Dawood [1] by proving that if \( p(z) \) has all its zeros in \(|z| \leq k, k \leq 1 \), then

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \left\{ \max_{|z|=1} |p(z)| + \frac{1}{k^{n-1}} \min_{|z|=k} |p(z)| \right\}. 
\] (1.9)

The result is best possible and equality holds for the polynomial \( p(z) = (z + k)^n \).

## 2 Main Results

In this paper we shall first present the following generalization as well as an improvement of (1.4) by considering the \( s \)-th derivative of \( P(z) \). More precisely, we have

**Theorem 1.** For \( p \in p_n \) and if \( p(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j} \), \( 1 \leq \mu \leq n \) having no zeros in \(|z| < k, k > 0 \), then for \( 0 < r \leq R \leq k, 1 \leq s \leq n \)

\[
\max_{|z|=R} |p^s(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1+k} \left( \frac{k^\mu + R^n}{k^\mu + r^n} \right)^{\frac{s}{n}} \left\{ \max_{|z|=r} |p(z)| - \min_{|z|=k} |p(z)| \right\}. 
\] (2.1)

The result is best possible for \( s = 1 \) and equality holds for \( p(z) = (z + k)^n \).

We now present some integral inequalities in the reverse direction for polynomials having a zero of order \( s \) at the origin. More precisely, we prove

**Theorem 2.** Let \( p \in p_n \) and \( p(z) \) having all its zeros in \(|z| \leq k, k \leq 1 \) with a zero of order \( s \) at \( z = 0 \), then for \( \beta \) with \(|\beta| < k^{n-s} \) and \( \gamma \geq 1 \)

\[
\left\| p' - \frac{\sin \gamma}{k^n \beta z^{s-1}} \right\|_\gamma \geq \begin{cases} 
\begin{aligned}
&n - (n - s)C_\gamma^n \| p + \frac{m}{k^n \beta z} \|_\gamma, \\
&\| p^{(s)} \|_\gamma 
\end{aligned}
\end{cases}
\] (2.2)

Where \( C_\gamma^n = \left\| \frac{k^\mu}{1+k^\mu z} \right\|_\gamma \).

By letting \( \gamma \to \infty \) in place of Theorem 2, we obtain

**Corollary 1.** Let \( p \in p_n \) and \( p(z) \) having all its zeros in \(|z| \leq k, k \leq 1 \) with a zero of order \( s \) at \( z = 0 \), then for \( \beta \) with \(|\beta| < k^{n-s} \)

\[
\left\| p'(z) + \frac{\sin \gamma}{k^n \beta z^{s-1}} \right\|_\infty \geq \begin{cases} 
\begin{aligned}
&\left( \frac{n + sk^\mu}{1+k^\mu} \right) \| p(z) + \frac{m}{k^n \beta z} \|_\infty, \\
&\| p^{(s)} \|_\infty 
\end{aligned}
\end{cases}
\] (2.3)

For \( \beta = 0 \) in inequality (2.3), we get

\[
\| p' \|_\infty \geq \left( \frac{n + sk^\mu}{1+k^\mu} \right) \| p \|_\infty 
\] (2.4)

Next, we prove the following result which yields refinements of both the inequalities (1.8) and (1.9) as special cases.

**Theorem 3.** If \( p(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j} \), \( 1 \leq \mu \leq n \) having all its zeros in \(|z| \leq k, k \leq 1 \), with \( s \)-fold zeros at the origin, \( 0 \leq s \leq n \), then

\[
\max_{|z|=1} |p'(z)| \geq \frac{(n + k^\mu s)}{(1+k^\mu)} \max_{|z|=1} |p(z)| + \frac{(n-s)}{k^{n-\mu}(1+k^\mu)} \min_{|z|=k} |p(z)|. 
\] (2.5)

The result is best possible and equality holds for \( p(z) = z^s (z + k)^{n-s} \), \( 0 \leq s \leq n \).

## 3 Lemmas

For the proof of these theorems, we need the following lemmas.

**Lemma 1.** If \( p(z) \) is a polynomial of degree \( n \) having no zeros in \(|z| < k, k \geq 1 \), then

\[
\max_{|z|=1} |p^s(z)| \leq \frac{n(n-1) \cdots (n-s+1)}{1+k} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right\}. 
\] (3.1)

Lemma 1 is due to Govil [7].

**Lemma 2.** If \( p(z) = a_0 + \sum_{j=\mu}^n a_j z^j \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having no zeros in \(|z| < k, k > 0 \), the for \( 0 \leq r \leq R \leq k \),

\[
\max_{|z|=R} |p(z)| \leq \left( \frac{k^n + R^n}{k^n + r^n} \right)^{\frac{n}{k}} \max_{|z|=1} |p(z)| + \left( 1 - \left( \frac{k^n + R^n}{k^n + r^n} \right)^{\frac{n}{k}} \right) \min_{|z|=k} |p(z)|. 
\] (3.2)

The above lemma is due to Dewan, Yadav and Pukhta [5].

**Lemma 3.** If \( p(z) = a_0 + \sum_{j=\mu}^n a_j z^j \), \( 1 \leq \mu \leq n \) is a polynomial of degree \( n \) having all its zeros in \(|z| \geq k \geq 1 \) and \( q(z) = z^n p\left( \frac{1}{z} \right) \), then for \(|z| = 1 \),

\[
k^\mu |p'(z)| \leq |q'(z)|. 
\] (3.3)

The above lemma is due to Chan and Malik [10]. By applying Lemma 3 to the polynomial one can deduce:

**Lemma 4.** If \( p(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j} \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having all its zeros in \(|z| \leq k \leq 1 \) and \( q(z) = z^n p\left( \frac{1}{z} \right) \), then for \(|z| = 1 \),

\[
k^\mu |p'(z)| \geq |q'(z)|. 
\] (3.4)

**Lemma 5.** Let \( p \in p_n \) and \( p(z) = a_0 + \sum_{j=\mu}^n a_j z^j \) having no zeros in \(|z| < k, k \geq 1 \), then for every complex number \( \beta \) with \(|\beta| \leq 1 \) and for each \( \gamma > 0 \),

\[
\left\| p'(z) + \frac{mn}{1+k^\mu \beta} \right\|_\gamma \leq C_\gamma^n \| p \|_\gamma. 
\] (3.5)
where \( C_γ^n = \left\| \frac{k^μ}{1 + k^μ z} \right\|_γ \).

The above lemma is due to Shah [11].

4 Proof of Theorems

Proof of Theorem 1. If \( p(z) \) has no zeros in \( |z| < k \), \( k > 0 \) and if \( 0 < r ≤ R ≤ k \), then \( G(z) = P(Rz) \) has no zeros in \( |z| < \frac{k}{R} \).

Thus applying Lemma 1 to \( G(z) \), we get

\[
\max_{|z|=1} |G^s(z)| ≤ \frac{n(n-1) \cdots (n-s+1)}{1 + \left( \frac{k}{R} \right)^s} \times \left\{ \max_{|z|=1} |G(z)| - \min_{|z|=\frac{1}{k}} |G(z)| \right\}
\]

which implies

\[
R^s \max_{|z|=1} |p^s(Rz)| ≤ \frac{n(n-1) \cdots (n-s+1)}{1 + \frac{k}{R}^s} \times \left\{ \max_{|z|=R} |p(z)| - \min_{|z|=k} |p(Rz)| \right\}
\]

which is equivalent to

\[
\max_{|z|=R} |p^s(z)| ≤ \frac{n(n-1) \cdots (n-s+1)}{R^s + k^s} \times \left\{ \max_{|z|=R} |p(z)| - \min_{|z|=k} |p(Rz)| \right\}
\]

(4.1)

Inequality (4.1) in conjunction with Lemma 2 yields,

\[
\max_{|z|=R} |p^s(z)| ≤ \frac{n(n-1) \cdots (n-s+1)}{R^s + k^s} \left( \frac{k^s + R^s}{k^s + k^s} \right) \times \left\{ \max_{|z|=R} |p(z)| - \min_{|z|=k} |p(Rz)| \right\}.
\]

This completes the proof of Theorem 1.

Proof of Theorem 2. We have \( p(z) = z^n \phi(z) \), where \( \phi(z) \) is a polynomial of degree \( n-s \), with the property that \( \phi(0) \neq 0 \), then

\[
q(z) = z^n \phi\left( \frac{1}{z} \right) = z^{n-s} \phi\left( \frac{1}{z} \right),
\]

is a polynomial of degree \( n-s \) with no zeros in \( |z| < \frac{1}{k} \).

Now if \( m_0 = \min_{|z|=\frac{1}{k}} |q(z)| = \frac{m}{k^n} \).

By Rouche’s theorem, the polynomial

\[
T(z) = q(z) + m_0 \beta z^{n-s}, \quad |\beta| < k^{n-s}
\]

of degree \( n-s \), will also have no zeros in \( |z| < \frac{1}{k} \).

Hence by Lemma 5, we have for \( \gamma ≥ 1 \) and \( |\beta| < k^{n-s} \)

\[
\|T\|_γ ≤ (n-s)C_γ^n \|T\|_γ
\]

or

\[
\left\| q'(z) + \frac{(n-s)m}{k^n} \beta z^{(n-s)-1} \right\|_γ ≤ (n-s)C_γ^n \left\| q(z) + \frac{m}{k^n} \beta z^{n-s} \right\|_γ
\]

i.e.,

\[
\left\| np(z) - nzp'(z) + \frac{(n-s)m}{k^n} \beta z^{s} \right\|_γ ≤ (n-s)C_γ^n \left\| p(z) + \frac{m}{k^n} \beta z^{s} \right\|_γ.
\]

Now by Minkowski inequality, we have for \( \gamma ≥ 1 \) and \( |\beta| < k^{n-s} \),

\[
n \left\| p(z) + \frac{m}{k^n} \beta z^{s} \right\|_γ ≤ \left\| np(z) - nzp'(z) + \frac{(n-s)m}{k^n} \beta z^{s} \right\|_γ + \left\| nzp'(z) + \frac{mS}{k^n} \beta z^{s} \right\|_γ
\]

which implies

\[
\left\| p'(z) + \frac{sm}{k^n} \beta z^{s-1} \right\|_γ \geq \left\{ n - (n-s)C_γ^n \left\| p(z) + \frac{m}{k^n} \beta z^{s} \right\|_γ \right\}.
\]

This completes the proof of Theorem 2.

Proof of Theorem 3. Since all the zeros of \( p(z) \) lie in \( |z| ≤ k \), with \( s \)-fold zeros at the origin, therefore, for every complex number \( \beta \) such that \( |\beta| < 1 \), it follows by Rouche’s theorem for \( m > 0 \) that the polynomial \( F(z) = p(z) - \frac{m}{k^n} \beta z^{s} \) has all its zeros in \( |z| ≤ k \), \( k ≤ 1 \). It can be easily verified that if

\[
H(z) = z^n F\left( \frac{1}{z} \right) = z^{n-s} G\left( \frac{1}{z} \right),
\]

\[
|H'(z)| = |nF(z) - zF'(z)|, \quad \text{for } |z| = 1.
\]

Applying Lemma 4 to the polynomial \( F(z) \), we get for \( |z| = 1 \),

\[
k^{1-s} |F'(z)| ≥ |H'(z)| = |nF(z) - zF'(z)|
\]

Again applying inequality (2.4) to the polynomial \( F(z) \), we get

\[
|F'(z)| ≥ \frac{n + sk^n}{1 + k^n} |F(z)|, \quad \text{for } |z| = 1.
\]

Replacing \( F(z) \) by \( p(z) - \frac{am}{k^n} z^n \) in (4.2), we get

\[
|p'(z) - \frac{am}{k^n} z^{n-1}| ≥ \frac{n + sk^n}{1 + k^n} \left| p(z) - \frac{am}{k^n} z^n \right|, \quad \text{for } |z| = 1
\]
for every \( \alpha \) with \( |\alpha| < 1 \). Choosing the argument of \( \alpha \) such that
\[
|p(z) - \frac{\alpha m}{k^n} z^n| = |p(z)| - |\alpha| \frac{m}{k^n}, \quad \text{for } |z| = 1.
\]
It follows from (3.3) that
\[
|p'(z)| - \frac{n|\alpha|m}{k^n} \geq \frac{n + sk\mu}{1 + k^n} \left\{ |p(z)| - \frac{m}{k^n} \right\},
\]
for \( |z| = 1 \).
Letting \(|\alpha| \to 1\), we obtain
\[
|p'(z)| \geq \frac{n + sk\mu}{1 + k^n} |p(z)| + \left\{ n - \frac{n + sk\mu}{1 + k^n} \right\} \frac{m}{k^n}
\]
this implies
\[
\max_{|z|=1} |p'(z)| \geq \frac{n + sk\mu}{1 + k^n} |p(z)| + \frac{(n - s)k}{k^n - k(1 + k^n) \min_{|z|=k} |p(z)|}.
\]
This completes the proof of Theorem 3. \( \square \)

References