A THEORY OF QUANTUM GRAVITY MAY NOT BE POSSIBLE BECAUSE QUANTUM MECHANICS VIOLATES THE EQUIVALENCE PRINCIPLE

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Abstract

Quantum mechanics (QM) clearly violates the weak equivalence principle (WEP). This implies that quantum mechanics also violates the strong equivalence principle (SEP), as shown in this paper. Therefore a consistent theory of quantum gravity (QG) that unifies general relativity and QM may not be possible unless it is not based upon the prevailing equivalence principle, or if quantum mechanics can change its mass dependence. Neither of these possibilities seems likely at the present time. Examination of QM in n-space, as well as relativistic QM does not change this conclusion.
1 Introduction

Following the 1930 pioneering work of Rosenfeld [1], all attempts to combine Einstein’s general relativity (EGR) and quantum mechanics into a theory of quantum gravity (QG) over the past seven decades have been futile – leading to discrepancies and even contradictions. A theory of QG may have far-reaching astrophysical implications. Quantum gravity could shed light on the big bang and the early universe. QG may also determine fundamental physical attributes that start on a small scale and affect large scale astrophysical properties. So it is important to explore a potential incompatibility. First let us examine the weak and the strong forms of the equivalence principle. It will be shown that the weak equivalence principle is clearly violated by both non-relativistic and relativistic quantum mechanics. The violation of the strong equivalence principle by quantum mechanics is more subtle, being difficult to show directly. However, it will be clearly shown that a violation of the weak equivalence principle implies a violation of the strong equivalence principle upon which EGR is based.

The 70 year failure to combine General Relativity and Quantum Mechanics into a consistent theory of quantum gravity is so old that two views have arisen. One is that the realms of the two theories are so different, that neither applies in the other; and hence they cannot be combined. The other is that it is meaningless to speak of quantum gravity and furthermore because there is no need to quantize gravity. My paper is a first step to consider the potential nature of the incompatibility, so that efforts can be mounted to resolve it.

2 Quantum Mechanics Clearly Violates the Weak Equivalence Principle

The weak equivalence principle (WEP) states that the trajectory of a freely falling particle in an external gravitational field is independent of its mass. This is related to the equivalence of inertial and gravitational mass which permits cancellation of the particle’s mass in Newtonian physics. Thus the same trajectory is followed by particles of different mass if they have the same initial conditions.
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However this does not occur in quantum mechanics. For a particle of inertial mass $m_i$ and gravitational mass $m_g$, falling directly toward mass M where $M \gg m$, the Schrödinger equation is 

$$-\frac{\hbar^2}{2m_i} \nabla^2 \Psi - \frac{G m_g M}{r} \Psi = i\hbar \frac{\partial \Psi}{\partial t}. \quad (1)$$

Even for $m_i = m_g$, the mass does not cancel out of the trajectory equations, as can be seen more transparently from eq. (2) since $m$ appears in the denominator of the kinetic energy term and the numerator of the potential energy term of the Hamiltonian. This is more apparent for $m_i = m_g \equiv m$, in a uniform gravitational field in the x direction, of acceleration $g$:

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi + mgx \Psi = i\hbar \frac{\partial \Psi}{\partial t}. \quad (2)$$

Thus the trajectory equations with expectation values of the acceleration, velocity, and radius are strong non-negligible functions of $m$ as shown by eqs. (5), (6), and (7).

In examining the possibility of gravitationally bound atoms in 3-space [2] and later in n-space [3, 4] it was clear to me that $m$ remains in the quantized trajectory equations, even for $M \gg m$; though $m$ cancels out of the classical trajectory equations in Newtonian gravity. One would expect $m$ to cancel out when averaging over states with large quantum numbers that puts them effectively in the classical continuum.

In quantum mechanics, the wavelength is inversely proportional to the momentum and hence involves the mass. Quantum mechanical interference effects in general, and quantum-gravitational interference effects in particular, depend on the phase and phase shifts which depend on the momentum and hence are proportional to the mass. For example for low energy, the phase shift $\delta_o \propto (k) = (p/\hbar) = mv/\hbar$. This is intrinsic to quantum mechanics. The uncertainty principle is basically related to the destruction of phase relations in different parts of the wave function. The weak equivalence principle requires that the free fall trajectory equations be independent of $m$. Even though the WEP and the SEP work well independently of QM, this is not the case in the union of general relativity and quantum mechanics (quantum gravity).
3 Quantum Mechanics Violates the Strong Equivalence Principle

Einstein postulated the strong equivalence principle (SEP) to formulate general relativity. The SEP states that locally, gravitation is indistinguishable from an equivalently accelerating reference frame. This implies the WEP since the accelerating reference frame is independent of the mass acted upon by the gravitational field. Furthermore, the SEP implies that it is possible to locally transform away gravitational effects in a properly chosen reference frame.

In the spirit of the SEP by transformation to an accelerated reference frame of acceleration \(-g\), with coordinates \(x_a\) in the Schrödinger equation (2), the transformed coordinates

\[
x_a = x - vt - \frac{1}{2}gt^2 \quad \text{and} \quad t = t_a
\]

do not involve the mass \(m\). So \(m\) doesn’t cancel out, and now both the WEP and the SEP are violated in this case. The violation of the WEP may be less manifest in the Heisenberg matrix quantum mechanics approach, say for example if the QM operator for a particle in a gravitational field were independent of mass. Nevertheless the trajectory equations would be mass dependent as the Heisenberg and Schrödinger approaches have been shown to be equivalent. We will soon see that quantum mechanics violates the WEP and hence the SEP quite generally.

In examining the possible violation of the SEP by quantum mechanics [4], varying modifications of Einstein’s SEP were presented. One of these is that of Rohrlich. Interestingly, Rohrlich [5] doesn’t use the terms WEP and SEP, and just refers to the equivalence principle (EP). Rohrlich’s general statement of EP says that the trajectory equations of a nonrotating test body in free fall in a gravitational field are independent of the energy content of that body. He appears to diminish the role of SEP in saying (p.42): ”True gravitational fields can never be transformed away. ... Apparent gravitational fields are a characteristic of the motion of the observer (rather than of the observed physical system), while true gravitational fields are the same for all observers no matter what their motion. ... Einstein’s statement
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[of the EP] as an equivalence between accelerated observers and gravitational fields is now seen to be restricted to apparent gravitational fields. True gravitational fields cannot be simulated by acceleration (i.e. by a coordinate transformation).” Although EGR has been applied to spinning bodies such as rotating black holes, the EP may not rigorously incorporate spin in general and particle spin in particular. Thus it should not be surprising if the Sagnac [6] effect cannot be properly accounted for within the context of EGR [7].

The different statements of the equivalence principle are inter-related. If the trajectory equations involve the mass, m, not only is WEP violated but this also involves a violation of SEP [4]. The mass not only does not cancel out but enters into the gravitational trajectory equations in a very non-negligible way.

It is clear that SEP ⇒ WEP since SEP yields trajectory equations of a body essentially independent of its mass in a gravitational field. My statement that the SEP implies the WEP is a shorthand way of saying that the SEP implies that the gravitational trajectory equations are independent or approximately independent of the mass of the orbiting body – as they are in Newtonian Gravitational physics.

A general principle in logic is:

\[ A \Rightarrow B \] then \[ \neg B \Rightarrow \neg A \].

Since SEP ⇒ WEP, then \[ \neg WEP \Rightarrow \neg SEP \].

Therefore quantum mechanics violates the SEP because it violates the WEP.

4 Semi-Classical Mechanics Violates the Weak Equivalence Principle

It may not be obvious that the semi-classical Bohr-Sommerfeld condition \[ \oint p \cdot dl = \oint mv \cdot dl = jh \] violates the WEP. Nevertheless, its predictions for the trajectory equations for a particle in a gravitational field clearly do so. We will see that they do so in all dimensions of space.

4.1 Three Dimensional Space

Because of the closer proximity and similarity of semi-classical mechanics to classical mechanics, one might think that its violation of
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the equivalence principle may not be as manifest as that of quantum mechanics. As we shall see, this is not the case.

For a two-body system of masses \( m << M \), the velocity of the orbiting body \( m \) is

\[
\frac{GmM}{r^2} = \frac{\mu v^2}{r} \approx \frac{mv^2}{r} \Rightarrow v = \left( \frac{GM}{r} \right)^{1/2},
\]

(3)

where circular motion is considered for simplicity, and the reduced mass \( \mu = \frac{mM}{m+M} \approx m \) for \( M >> m \). The Bohr-Sommerfeld condition for the angular momentum leads to

\[
m(v)r = m \left( \frac{GM}{r} \right)^{1/2} \Rightarrow r = j\hbar,
\]

(4)

where \( j \) is an integer. Equation (4) implies that the orbital radius is

\[
r = \frac{GMm}{j\hbar}.
\]

(5)

Substituting eq. (5) into eq. (3) we obtain the velocity

\[
r = \frac{j^2\hbar^2}{GMm^2}.
\]

(6)

and the acceleration is

\[
a \equiv \frac{v^2}{r} = -G^3M^3m^4/(j\hbar)^4.
\]

(7)

These all have a mass dependence, \( m \), which is not present in their classical counterparts.

4.2 n-Dimensional Space

Sometimes one encounters surprises in higher n-dimensional space [3, 4]: Angular momentum cannot be quantized in the usual manner in four dimensional space. This is because the dependence of angular momentum, \( L \), on \( r_n \) allows the orbital radius to adjust in the quantization of \( L \) in all dimensions except in 4-space. There is no binding energy for atoms for \( \geq 4 \)-space because the binding energy \( = 0 \) for \( n = 4 \), and the energy levels are all \( > 0 \) for \( n > 4 \). The macroscopic dimensionality of space can be determined by measuring the temperature dependence of black body radiation because the
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temperature exponent = n + 1. The volume of a radius r, infinite
dimensional sphere = 0 because the n-volume of an n-sphere relative
to an n-cube of side = r, peaks \( \approx 5 \)-dimensional space. Thereafter for
large n, the ratio of the n-sphere volume to the n-cube volume is a
quickly diminishing fraction which \( \to 0 \) as \( n \to \infty \). So it is important
to look at higher dimensions in case one might encounter a surprise
with respect to the WEP.

However, it is easy to demonstrate that semi-classical mechanics
violates the WEP in n-dimensional space. Because of the correspon-
dence of the m dependence in quantum mechanics, QM violates the
WEP in n-space. We can directly obtain the higher dimensional re-
sults for circular orbits directly from [3, 4]:

For \( M >> m \), equating the gravitational force and the centripetal
force

\[
F_n = \frac{-2\pi G_n M m \Gamma \left( \frac{n}{2} \right)}{r^n} = -\frac{mv_n^2}{r_n},
\]

where the n-space universal gravitational constant \( G_n \) changes, in a
way that is model dependent, from its 3-space value. The Gamma
function \( \Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt \) for all n (integer and non-integer).
When n is an integer, \( \Gamma(n) = (n-1)! \) \( \hbar \) is (Planck’s constant)/2\( \pi \).
Combining eq. (8) and the Bohr-Sommerfeld condition, \( mv_n r_n = j\hbar \),
we find for the orbital radius of m around M

\[
r_n = \left[ \frac{j\hbar \pi^{n/2}}{m \left[ 2G_n M \Gamma \left( \frac{n}{2} \right) \right]^{1/2}} \right]^{\frac{2}{4-n}}.
\]

In 3-space eq. (9) yields \( r_3 = j^2 \hbar^2/3Mm^2 \), the same as in eq. (5).
Similarly, the orbital velocity is

\[
v_n = \left\{ \left[ \frac{2\pi G_n M \Gamma \left( \frac{n}{2} \right)}{\pi^{n/2}} \right] \left[ \frac{m^{2/(4-n)} \left[ 2G_n M \Gamma \left( \frac{n}{2} \right) \right]^{1/(4-n)}}{(j\hbar)^{2/(4-n)} \pi^{(n-2)/2(4-n)}} \right] \right\}^{1/2}.
\]

In 3-dimensions eq. (10) gives \( \pi \), the same result as eq. (6).

The acceleration of the orbiting mass m is

\[
a_n = \frac{F_n}{m} = -\frac{2\pi G_n M \Gamma \left( \frac{n}{2} \right)}{\pi^{n/2}} \left[ \frac{m \left[ 2G_n M \Gamma \left( \frac{n}{2} \right) \right]^{1/2}}{j\hbar \pi^{(n-2)/4}} \right]^{\frac{2n-2}{4-n}}.
\]
In 3-dimensional space, eq. (11) yields $a_3 = -G^3 M^3 m^4 / (j\hbar)^4$, the same as eq. (7).

It is clear that the acceleration, the orbital radius, and orbital velocity are all functions of the mass $m$ in all dimensions as a result of quantization. The presence of $m$ is not an artifact of the Bohr-Sommerfeld condition. The same mass dependency and basically the same results are obtained from the Schrödinger equation. (The Hamiltonian for the attractive gravitational potential is of the same form as that of the hydrogen atom with radial wave function solutions in terms of Laguerre polynomials. In a one dimensional uniform gravitational potential, the Schrödinger equation can be solved in terms of Airy functions, Bessel functions of order 1/3, or equivalently MacDonald functions.) The failure of $m$ to vanish indicates that quantum mechanics is inconsistent with the weak equivalence principle [2, 3].

The above results indicate that quantum mechanics also violates the WEP in higher dimensional $n$-space. By the same principle of logic used in Sec. 3, one may conclude that the violation of WEP in $n$-space implies the violation of SEP in $n$-space. Thus quantum mechanics violates the SEP in all space and is incompatible with Einstein’s general relativity in all space. It is noteworthy that these semi-classical results for gravitational orbits do not approach the classical results as $j \to \infty$ or as $\hbar \to 0$, whereas the quantum mechanical results may have the ability to do so because of additional degrees of freedom in combining wave functions.

5 Klein-Gordon and Dirac Relativistic Quantum Mechanics Violate the WEP

The special relativity Klein-Gordon and Dirac equations reduce to the Schrödinger equation in the non-relativistic limit, so their predictions must agree with it in this limit. Thus in the non-relativistic limit, they must give trajectory equations with expectation values of the acceleration, velocity, and radius that are a function of the particle mass $m$ freely falling in a gravitational field, and hence violate the WEP. We shall see that they also do not obey the WEP in the relativistic limit.

The SEP for a non-uniform gravitational field applies at a point, or at best in a very small localized region of space. Therefore in gen-
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eral, the SEP is precluded from applying to QM which is inherently non-local. Since the WEP does not require locality, let us examine these relativistic quantum equations for a free particle to ascertain if they can escape violation of the WEP when a gravitational field is included in the Hamiltonian. Furthermore, a question may be raised about a potential energy representation of gravity since gravity is just space-time curvature in EGR. We avoid the unnecessary complications of a proper depiction of the gravitational source (which in the case of EGR is an energy-momentum tensor that is typically taken to be a kinematic perfect fluid) by considering the free-particle equations.

The Schroedinger equation is inherently not relativistic because it doesn’t treat space and time on an equal footing. Its space derivatives are second order and its time derivative is first order. Dirac circumvented this dilemma by modifying the Hamiltonian so that it would be linear in the space derivatives. Thus the Dirac equation for a free mass, \( m \), is

\[
\left[ i\hbar \frac{\partial}{\partial t} - i\hbar c \alpha \cdot \nabla + \beta mc^2 \right] \Psi = 0, \tag{12}
\]

where the components of the vector \( \alpha \), and the scalar \( \beta \) are Dirac matrices. \( \Psi \) is a four-component spinor field. (Incidently, although the Dirac equation is only for special relativity, spinors were discovered by Cartan while working on general relativity.) The matrices \( \alpha \) and \( \beta \) are independent of \( m \), as are the first two terms, and \( \beta mc^2 \Psi \) is the only mass term. If one were to introduce a special relativistic gravitational potential energy term in the Hamiltonian, it would be proportional to \( m \), and \( m \) would not cancel out. Therefore the Dirac equation violates the WEP, implying that it also violates the SEP, as well as due to non-locality.

The Klein-Gordon equation for a free mass, \( m \), is

\[
\left[ \hbar^2 \frac{\partial^2}{\partial t^2} - \hbar^2 c^2 \nabla^2 + m^2 c^4 \right] \Psi = 0. \tag{13}
\]

In addition to a second-order time derivative and a second-order space derivative, the Klein-Gordon equation has a term \( m^2 c^4 \Psi \) which is the only mass term. If one were to introduce a special relativistic gravitational potential energy term in the Hamiltonian, it would be
proportional to m, and m would not cancel out. Therefore the Klein-Gordon equation violates the WEP, implying that it also violates the SEP, as well as due to non-locality.

6 Discussion

Quantum mechanics violation of the WEP and hence the SEP is a fundamental deterrent to the unification of EGR and QM for the creation of a theory of quantum gravity. It is much more fundamental in the context of this quest, than violations and anomalies in other branches of physics. For example, the radiation reaction force gives runaway solutions in which a free charge increases its energy without limit. Although this violates conservation of energy which electrodynamics obeys, it is not a deterrent to an otherwise consistent theory of electrodynamics. [Interestingly, when the radiation reaction force is included in the classical theory of atomic orbits, as an electron radiates energy away it does not fall into the atomic nucleus, contrary to common consensus. Instead as it approaches the nucleus, the radiation reaction force (proportional to the time rate of acceleration) becomes sufficiently large to propel it back out well beyond its initial orbit in violation of energy conservation.] The argument that QM violates the WEP and hence the SEP may prevent a theory of quantum gravity is more akin to the proof that irrational numbers like the $\sqrt{2}$ cannot be written as fractions.

To my knowledge, my approach to the problem preventing the development of a theory of quantum gravity is original and differs from that of others. Loinger [8] comes to the same conclusion as mine. However, he takes a different approach in reaching his conclusion. One difficulty that has been previously examined is the different role that time plays in Einstein’s general relativity compared with quantum mechanics. Time is an external scalar parameter in quantum mechanics. In EGR time is part of space-time, and hence is an internal dynamical quantity. It is difficult to reconcile the two concepts. Unruh [9] aptly discusses this in his treatise on “Time, Gravity, and Quantum Mechanics”. Herdegen and Wawrzycki [10] in their paper, “Is Einstein’s equivalence principle valid for a quantum particle?” conclude that the EP and quantum mechanics are compatible. Davies [11] discusses tunneling anomalies related to “Quantum mechanics and the equivalence principle.” Although he finds that QM
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violates the WEP, he does not seem to find an incompatibility between quantum mechanics and the SEP. Schiff [12] conjectured (but did not prove) that the WEP implies the SEP. As far as I could ascertain, neither Schiff nor anyone else has commented on whether or not the SEP implies the WEP and the implications of this.

7 Conclusion

Quantum mechanics clearly violates the WEP, and logic shows that a violation of the WEP implies a violation of the SEP. This is the case for n-dimensional space, as well as for non-relativistic and relativistic QM. Therefore to achieve a consistent theory of quantum gravity, it must not be based upon the prevailing equivalence principle. Or quantum mechanics must change a body’s mass dependence in the trajectory equations of that body in a gravitational field. Neither of these seems likely at the present time.

As difficult as the reconciliation of the differences of the concepts of time and non-locality in QM with EGR, this appears much easier than the resolution of QM’s violation of the equivalence principle. The equivalence principle is at the heart of EGR, yielding a deep connection between spacetime curvature and gravity. Thus, unless QM can be made to comply with the equivalence principle (EP) or a modification of it, the union of quantum mechanics with Einstein’s general relativity seems doubtful.

References
Comment on

A THEORY OF QUANTUM GRAVITY MAY NOT BE POSSIBLE BECAUSE QUANTUM MECHANICS VIOLATES THE EQUIVALENCE PRINCIPLE

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Let me make the general remark that the paper impress informed reader as if was written by an amateur of physics rather than by physicist. This impression emerges already from the title. Assertion that a nonexisting theory may or may not be possible fits oddly with the scientific methodology of physics. Instead of giving the whole list of author’s naiveté let me just present the key part of his paper, namely his ”proof” that the strong equivalence principle (SEP) implies the weak (WEP). The author defines the WEP as the requirement on equations of motion (irrespectively if they are ordinary differential equations as for pointmass particle or partial differential equations as for Schroedinger’s equation) containing a parameter which we may on the other hand interprete as mass to be independent of the mass. The SEP states that it is possible to locally transform away gravitational effects by a proper choice of reference frame (this vague
The paper is not appropriate for any serious scientific journal especially it is not appropriate for Concepts of Physics.
Reply to referee’s comments.

1. Reviewer: "written by an amateur of physics rather than by physicist."


2. Reviewer: "Assertion that a non-existing theory [MR: in spite of a 70 year effort] may or may not be possible fits oddly with the scientific methodology of physics."

On the contrary, physics and all of science progresses by showing that certain potential theories or approaches are flawed. This is the case for the flat earth theory, the geocentric theory of Ptolomey, the phlogiston theory, the attempts to show that the square root of 2 is rational, etc.

3. Reviewer: "The author applies 'SEP implies WEP' to the Schroedinger equation –exactly as for Schroedinger or Dirac equation –the mass does not cancel out even if the inertial mass is the same as the gravitational mass."

I’m glad that the Reviewer agrees with me that the mass does not cancel out for the Schroedinger or Dirac equations, i.e., it is not just a non-relativistic quantum mechanical manifestation. Rather it is intrinsic to all of existing quantum mechanics. As shown in my paper, the mass not only does not cancel out but enters into the gravitational trajectory equations of acceleration, velocity, and radius in a very non-negligible way.

4. Reviewer: "If the author’s implication 'SEP implies WEP” was
true the author would have to conclude that general relativity contra-
dicts SEP.

My statement that the SEP implies the WEP is a shorthand way of saying that the SEP implies that the gravitational trajectory equations are independent or approximately independent of the mass of the orbiting body - as they are in Newtonian Gravitational physics. It does not challenge or intend to challenge General Relativity.

Saying that the SEP implies the WEP no more says that general relativity contradicts the SEP than does the 1960 Schiff Conjecture that the WEP implies the SEP.

5. Author’s General Comments: The 70 year failure to combine General Relativity and Quantum Mechanics into a consistent theory of quantum gravity is so old that two schools of thought have arisen. One is that the realms of the two theories are so different, that neither applies in the other; and hence they cannot be combined. The other is that it is meaningless to speak of quantum gravity not only because the two concepts will always be incompatible for unknown reasons, but because there is no need to quantize gravity. My paper is a first step to consider the potential nature of the incompatibility, so that efforts can be mounted to resolve it.

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